AN

ELEMENTARY TREATISE

ON

MECHANICS.

BY

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PREFACE.

The application of the higher branches of mathematical analysis to the solution of mechanical problems has been so perfectly successful as to procure its universal adoption, not only in the treatment of abstruse and difficult questions, but also of the most simple and elementary parts of the science.

The effect of this has been to reduce the entire subject to an uniform system, but at the same time to place all parts of it alike beyond the reach of the student who is acquainted only with the elements of geometry.

The difficulty of finding a treatise on Mechanics free from this inconvenience has long been felt in the University of Dublin, and has been brought more immediately under the notice of the Author by his connexion with the School of Engineering. In the hope of contributing to its removal, he has been induced to publish the following treatise, in which he has endeavoured, by means of simple geometrical constructions, to render the most important fundamental propositions easily understood by all classes of students.

In the Notes a brief outline is given of the method of applying to mechanical science the principles of Algebra, and of the Calculus; but the Author feels that it is unnecessary to dwell long on this application, as he could not hope to improve on the manner in which it has been already treated in Lloyd's Mechanical Philosophy, Venturoli's Theory of Mechanics, and Poisson Traité de Mécanique.

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AN ELEMENTARY TREATISE

ON

MECHANICS.

INTRODUCTION.

Force is the general name for whatever tends to produce motion in a body, whether it does actually produce any motion, or is counteracted by the opposite tendency of other forces.

The direction of a force is the direction of the motion which it tends to produce in a body previously at rest.

The object of the science of *Mechanics* is to determine the effect of forces; it is divided into two parts—*Statics*, or the science of equilibrium; and *Dynamics*, or the science of motion: the object of the former being to determine under what conditions forces will counteract one another, or be in equilibrio, and the object of the latter to ascertain the motion which will result when these conditions are not fulfilled.

Some authors have derived the conditions of equilibrium from the laws of motion; but the more methodical and satisfactory course of treating the subject, is to separate these two parts of mechanics, and demonstrate the first, without any reference to those laws which more properly belong to the second. For this purpose it is necessary to lay down the following definitions and principles.

PART I.

STATICS.

CHAPTER I.

ON THE COMPOSITION AND RESOLUTION OF FORCES.

DEFINITION 1. Two forces are said to be equal, if acting on the same point in opposite directions, they produce no effect.

2. One force is said to be equal to the sum of two or more other forces, if it produces the same effect that the others would produce if they acted together in the same direction.

3. The ratio of two forces is the ratio of the numbers of equal forces of which they are the sums.

4. A rigid body is one whose parts always preserve the same distances from one another.

5. Forces are represented by right lines, if these lines are in the directions of the forces and proportional to them. These lines are sometimes said to be equal to the forces which they represent.

Axiom 1. The direction of a force is always a right line.

2. A force acting in a given direction on any point in a rigid body, produces the same effect as if it acted on any other point in the body in the line of its direction.

Cor. Any force acting on a rigid body is equilibrated by the resistance of a fixed point in the right line of its direction; for, by the preceding principle, the force will produce the same effect as if it was applied to the fixed point; and as this point cannot move, the force is incapable of producing motion.

3. Any two forces acting in different directions on the same point, are equivalent to a single force which is called their

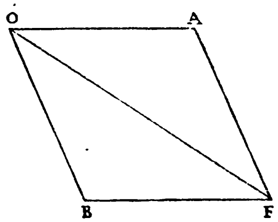
resultant, and whose direction lies between the directions of the two forces; the only possible tendency of the two forces being to produce motion, and the point on which they act being only capable of moving in one direction at one time, the tendency of the two forces is precisely the same as of a single force in that direction; and it is evident, that it must be intermediate between the two directions in which either force by itself would cause the body to move.

4. The resultant of two equal forces acting on the same point, is in the direction of the bisector of the angle contained by the directions of the forces.

5. Two forces cannot make equilibrium, unless they are equal and directly opposed to each other.

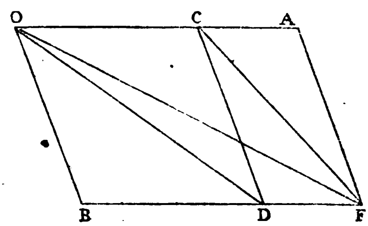
6. A single force cannot be equilibrated by the resistance of a fixed point, unless its direction passes through that point.

7. If a body be kept at rest by any number of forces, the equilibrium will not be disturbed by fixing any point in the body.



Prop. 1. If two forces acting on the same point O, are represented in quantity and direction by the right lines OA, OB, their resultant will be in the direction of OF, the diagonal of the parallelogram OAFB.

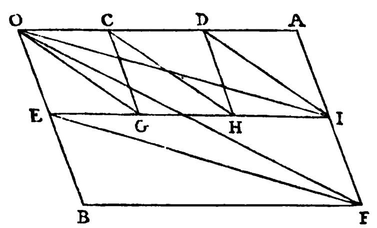
1. Let the right lines OA, OB be equal, then the forces are equal, and their resultant bisects the angle AOB (Ax. 4); but since AF is equal to OB, it is also equal to OA, therefore the angle AOF is equal to AFO, which is equal to FOB, therefore OF bisects the angle AOB, and therefore it is the direction of the resultant.



2. Let the force OA be equal to the sum of two forces OC and CA, such that the resultant of OC and OB is in the direction OD, and that the resultant of CA and

CD is in the direction CF, then the resultant of OA and OB will be in the direction OF. For, suppose the point F to be fixed, the force OA produces the same effect as the two forces OC and CA acting in the same direction, OC acting at the point O and CA at the point C (Ax. 2); but OC and OB together produce a resultant in the direction OD (by hypothesis), therefore the given forces OA and OB are equivalent to this resultant and CA, but the resultant in the direction OD may be applied at D without altering its effect (Ax. 2); and since it was equivalent to the two forces OC, OB, it must also be equivalent to the two forces BD, CD, which are equal and parallel to them; consequently, the given forces are equivalent to the three forces BD, CD, and CA; but the force BD, whose direction passes through the fixed point F, is equilibrated by its resistance, and the only remaining forces are CD and CA, which may both be supposed to be applied at the point C, and are (by hypothesis) equivalent to a resultant in the direction CF, therefore these forces are also equilibrated by the resistance of the fixed point F; therefore, the whole effect of the forces OB, OA, is equilibrated by the resistance of the fixed point F, and therefore their resultant passes through that point (Ax. 6).

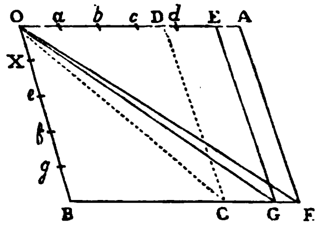
3. Let the lines which represent the two forces be commensurable.



Let OC be the common measure of OA and OB, and let OC, CD, DA, OE, EB, be equal to one another; then since OC and OE are equal, their resultant is in the direction OG (Case 1), and since CD and CG are equal, their resultant is in the direction CH (Case 1); therefore, the resultant of OD and OE is in the direction OH (Case 2); and since DA and DH are equal, their resultant is in the direction DI (Case 1), therefore, the resultant of OE and OA is in the direction OI (Case 2); in like

manner it can be shewn, that the resultant of EB and EI is in the direction EF, therefore (Case 2), the resultant of OA and OB is in the direction OF, and the same may be shewn in like manner if OC were any other common submultiple of OA and OB; therefore, if these lines are commensurable, the resultant will be in the direction of the diagonal OF.

4. Let the lines OA and OB be incommensurable, the resultant will be in the direction of the diagonal OF.

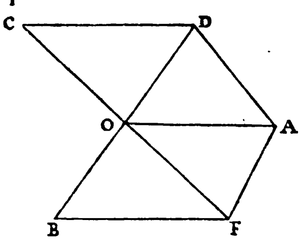


For, if it be possible, let it be in any other direction OC, draw CD parallel to OB, and let OX, a submultiple of OB, be taken less than AD; and let parts be taken as often as possible on OA equal to OX, the remainder EA will be less than OX, therefore, *à fortiori*, less than DA, therefore E lies between D and A; but since the right lines OE and OB are commensurable, the resultant of forces represented by these lines is in the direction of OG, the diagonal of the parallelogram OEGB (Case 3); and since the given forces OA and OB are equivalent to these forces OB and OE, together with the force EA, they are therefore equivalent to the force EA, and a force in the direction OG; but the direction of the resultant of these two forces lies within the angle AOG, contained by the directions of the forces (Ax. 3), and therefore, cannot be in the direction OC, as was supposed; in like manner it may be shewn, that this resultant is not in any other direction except OF, therefore OF is its direction.

Thus it is proved, that whether the lines OA and OB are commensurable or incommensurable, the resultant of the forces represented by these lines is always in the direction of the diagonal OF, which was to be proved.

Note. The above proof is derived from the principles usually adopted in Statics; but the same may be shewn much more simply, from considerations derived from the laws of

motion; for it is a well-known fact, that if two forces be applied simultaneously to a point, its motion in a given time will be compounded of the motions which the two forces would have produced separately in the same time, and these motions are proportional to the forces, therefore will be represented by the lines OA and OB; but the motion compounded of motions from O to A and from O to B is a motion from O to F (for this compound motion is the motion which would be produced in a body moving from O to A, while the line OA is moved in a direction parallel to itself from the position OA to BF; so that, when the body reaches A, the point A will have reached F), and this is the motion which would have been produced by a single force in the direction OF, therefore OF is the direction of the resultant of OA and OB; this is nearly the manner in which Sir Isaac Newton (the inventor of the theory of the composition of forces) proved this proposition.

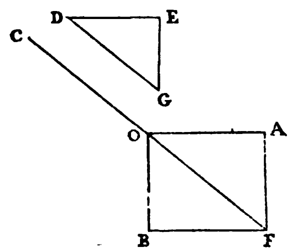


Prop. 2. The resultant of two forces represented in quantity and direction by the right lines OA, OB, is represented both in quantity and direction by the diagonal OF of the parallelogram OAFB.

For the resultant is in the direction OF (Prop. 1); produce FO to C, and let OC be equal to the resultant of OA and OB; then since it is also directly opposite to this resultant, the three forces OA, OB, OC, will make equilibrium, therefore the resultant of the two forces OA and OC is equal and opposite to OB; but this resultant is in the direction of the diagonal OD of the parallelogram OCDA, therefore OD and OB are directly opposite, that is, they are in one right line, therefore OD is parallel to AF, but AD is also parallel to OF, therefore ADOF is a parallelogram, and OF is equal to AD; but since OCDA is also a parallelogram, OC is equal to AD, therefore

OF is equal to OC; but OC was made equal to the resultant of OA and OB, therefore OF is equal to this resultant, and it is also in the same direction.

Prop. 3. If three forces acting on the same point make equilibrium, and if any three right lines, DE, EG, GD, parallel to their directions, form a triangle, the sides of this triangle are proportional to the forces.

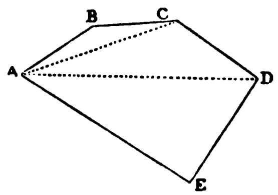


Let the forces be represented by the right lines OA, OB, OC, and since they make equilibrium, any one of them OC is equal and opposite to the resultant of the other two, complete the parallelogram OAFB, and the diagonal OF will represent this resultant in quantity and direction (Prop. 2), therefore it is equal to, and *in directum* with OC; but AF is also equal to OB, therefore the sides of the triangle OAF are equal to the lines which represent the forces; but since the sides of the triangle DEG are respectively parallel to the sides of OAF, these triangles are similar, therefore the sides DE, EG, GD, are proportional to OA, AF, OF, or to OA, OB, OC.

Cor. If the three forces be perpendicular to the three sides of a triangle, they are proportional to them, for this triangle is equiangular to DEG, and therefore similar to it.

Note. Since the sides of the triangle are proportional to the sines of the opposite angles, the three forces which make equilibrium are proportional each to the sine of the angle contained by the other two; also, if the directions of two forces which act on the same point, form a right angle, the square of their resultant is equal to the sum of their squares, for it is represented by the hypothenuse of a right-angled triangle, of which they are the sides; also the cosine of the angle which it makes with either of the component forces is to radius as that force is to the resultant, as is evident.

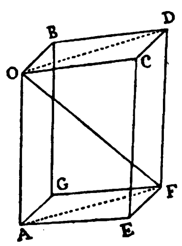
Prop. 4. If any number of forces act upon a point, and if right lines AB, BC, CD, DE, be drawn parallel and equal to those which represent these forces, the right line AE will be parallel and equal to that which represents their resultant, provided that all the lines AB, BC, &c., be drawn in the directions in which the forces act and not in the opposite directions.



For the resultant of two forces equal and parallel to AB and BC, is itself equal and parallel to AC, and the resultant of this force and one equal and parallel to CD, is equal and parallel to AD, and the resultant of this last resultant and another force equal and parallel to DE, is equal and parallel to AE, therefore, AE is equal and parallel to the resultant of the given forces. It is evident, that the same could in like manner be proved of any other number of forces.

Cor. If the point E coincide with A, the line AE will vanish, or there will be no resultant; that is to say, any number of forces acting on a point will be in equilibrio, if they be proportional to the sides of a complete polygon, to which they are respectively parallel, the directions of the forces being in the directions of these sides taken in order.

Prop. 5. If three forces OA, OB, OC, not in the same plane, act on the same point O, their resultant will be in the direction of the diagonal OF of a parallelopiped, of which OA, OB and OC are the edges.

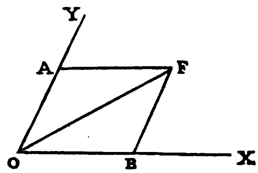


For join OD and AF, then since OA and DF are equal and parallel to CE, they are equal and parallel to one another, and the figure OAFD is a parallelogram, and its diagonal OF is the resultant of the forces OA and OD; but OD is the resultant of OB and OC, therefore, OF is the resultant of OA, OB, and OC.

Prop. 6. To resolve a given force OF into two others in

given directions OX and OY, which are in the same plane with OF, and intersect it at the same point O.

Through F draw right lines, FA and FB, parallel to OX and OY, and forming the parallelogram OAFB, then OA and OB are the required forces, for they are in the directions OX and OY, and OF is their resultant (Prop. 2).

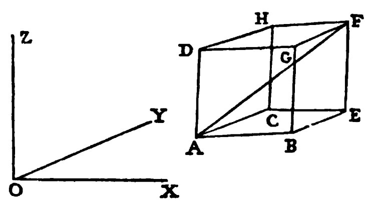


Cor. 1. Hence, if two points be fixed in a rigid body, their resistance will make equilibrium with any force whose direction is in a plane containing these points; for it may be resolved into two forces which pass through the two points, and are therefore equilibrated by their resistance.

Cor. 2. If the force OF be represented by P, and the angles FOX and FOY by α and β, and if XOY be a right angle, the components in the directions OX and OY will be P $\cos \alpha$ and P $\cos \beta$.

Prop. 7. To resolve any given force AF into three forces parallel to three given lines, OX, OY, OZ, which do not lie in the same plane.

Through A draw three lines AB, AC, AD, parallel to the three given lines, and through F draw FE parallel to AD or OZ, and meeting the plane BAC in E, through E draw EB and EC parallel to AC and AB, and complete the parallelopiped ABCDF, then will AB, AC, and AD be the required forces, for they are in directions parallel to the given lines, and AF, being the diagonal of the parallelopiped, is their resultant (Prop. 5).

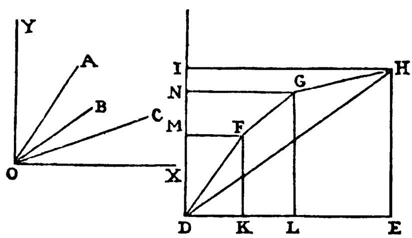


Cor. 1. Hence, if three points in a rigid body (not in the same right line) be fixed, any force which acts on the body will be equilibrated, for if the direction of the force be in the

plane of the three points, there will be equilibrium (Cor. Prop. 6), and if it be not in that plane, right lines may be drawn from a point in its direction to the three fixed points, which will not be in the same plane; therefore the given force may be resolved into three, in the directions of these three lines, and they will evidently be equilibrated by the resistance of the fixed points.

Cor. 2. If the angles XOY, XOZ, YOZ be right angles, and if the force AF be represented by P, and the angles which it makes with AB, AC, AD, by α, β, γ, the components in these directions will be $P \cos \alpha$, $P \cos \beta$, $P \cos \gamma$.

Prop. 8. If any number of forces OA, OB, OC, acting in the same plane, on a point O, be resolved in the directions OX, OY, and if their resultant be resolved in the same directions, each component of the resultant is equal to the sum of the components of all the separate forces in the same direction.



For, draw lines DF, FG, GH, equal and parallel to OA, OB, OC, then DH is equal and parallel to their resultant (Prop. 4); and if DI and DE be drawn parallel to OY and OX, and the parallelogram DIHE completed, then DI and DE will be equal to the components of the resultant in the given directions (Prop. 6); and if FM and GN be drawn parallel to OX, and FK and GL parallel to OY, then DM and DK, MN and KL, NI and LE, will be evidently equal to the components of the separate forces in the same directions; but DI is equal to the sum of DM, MN, and NI, and DE is equal to the sum of DK, KL, and LE.

Cor. 1. In this proposition it is supposed, that all the components are directed *from* the point O; but if any of these

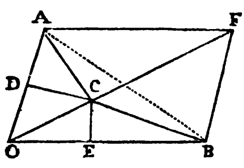
components are directed *to* this point, it is evident that, in forming the sum, such components should be subtracted and not added.

Cor. 2. If the forces are not in the same plane, each of them may be resolved into three parallel to three right lines (Prop. 7), and it may be proved as above, that if the resultant be resolved into three parallel to the same three lines, each of these will be equal to the sum of the components of the separate forces in the same direction, observing as before, that in the formation of these sums, any force which acts in a direction opposite to the rest, should be subtracted and not added.

Definition 6. The moment of a force with regard to a given point is the product of that force multiplied by the perpendicular let fall on it from the given point.

Definition 7. Forces are said to act in opposite directions with regard to a given point, if that point be on the right hand side of the direction of one force, and on the left hand side of the direction of the other; and they are said to act in the same direction with regard to a point, if it be on the same side of the direction of each of the forces.

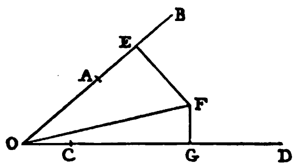
Prop. 9. If two forces OA, OB act on a point O, and if their moments be equal and their directions opposite with regard to another point C in the plane AOB, that point is on their resultant.



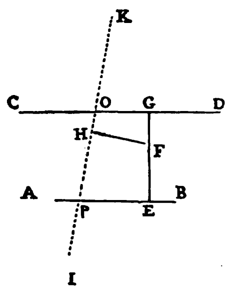
Let fall the perpendiculars CD, CE, and join CA, CB, and AB, then since the forces OA and OB act in opposite directions with regard to the point C, that point must lie within the angle AOB (Def. 7); but the rectangle under CD and AO is the moment of the force OA, and the rectangle under CE and OB is the moment of OB (Def. 6), therefore these rectangles are equal to one another; and therefore the triangles OCA, OCB, which are their halves, are also equal, but they stand on the same base CO, therefore

their altitudes are equal, and the right line AB, which joins their vertices, is bisected by the base CO; but AB is one diagonal of the parallelogram OAFB, therefore CO, which bisects it, must coincide with the other diagonal OF, which is the resultant of the given forces OA and OB (Prop. 1).

Prop. 10. If two forces AB, CD act in the same plane, on a rigid body containing a fixed point F in that plane, and if their moments are equal and their directions opposite with regard to that point, there will be equilibrium.



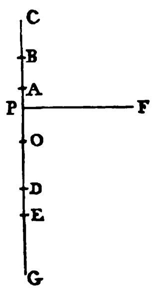
1. If the forces AB, CD be not parallel, let them meet in O, and join OF, then since these forces produce the same effect as if they acted at O (Ax. 2), and since their moments are equal, and their directions opposite with regard to F, the point F is on their resultant (Prop. 9), and therefore, since it is fixed, there is equilibrium.



2. If AB and CD be parallel, from F draw any line FH equal to FE, and not perpendicular to AB, through H draw OP perpendicular to FH, and meeting AB and CD, in O and P, take parts on this line, HK and HI, each equal to AB, and let forces represented in quantity and direction by these lines act on the given body; then, since FH and HK are equal to FE and AB, the moment of HK is equal to the moment of AB with regard to F, and in like manner the moment of HI is equal to the moment of AB with regard to F; but the moment of CD was also supposed equal to the moment of AB with regard to F, therefore the moments of these four forces with regard to F are equal to one another; but since HK and HI are in opposite directions, one of them, HK, must be in a direction opposite to AB with regard to the

fixed point F, and the other HI in a direction opposite to CD with regard to the same point, and therefore since CD and HK meet in O, they make equilibrium (Case 1); and since AB and HI meet in P, they also make equilibrium; therefore there is equilibrium between the four forces AB, CD, HI, HK, but the two latter being equal and opposite produce no effect (Def. 1), therefore the two former are in equilibrio.

Prop. 11. If any number of forces, whose directions are in the same plane, make equilibrium, the sum of the moments of those that act in one direction with regard to any point F in that plane, is equal to the sum of the moments of those that act in the opposite direction with regard to the same point.



1. If the forces act in the same right line CG, it is necessary for equilibrium, that the sum of OA, AB, BC, which act in one direction, be equal to the sum of OD, DE, EG, which act in the opposite direction, and therefore, if each of these sums be multiplied by the perpendicular FP, the products, which are the sums of the moments with regard to F, will be equal.

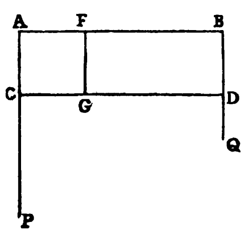
2. If the forces act in different right lines, let the point F be fixed, and there will still be equilibrium (Ax. 7); but if forces OA, AB, BC, be applied to the body in the right line CG, whose moments with regard to F are respectively equal and opposite to the moments of each of the given forces which act in one direction with regard to that point; and if forces OD, DE, EG, be applied in the same right line, whose moments are equal and opposite to those of the other given forces, there will be equilibrium between each pair of these forces (Prop. 10); therefore since the given forces make equilibrium (by hypothesis), the forces in the right line CG must also make equilibrium, and therefore the sum of the moments of OA, AB, BC, in one direction, is equal to the sum of the mo-

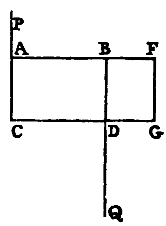
ments of OD, DE, EG, in the opposite direction (Case 1); but these moments are respectively equal to the moments of the given forces, therefore their sums are also equal.

Cor. 1. If there be any forces in the same plane, and not in equilibrio, and if the sum of the moments of those that act in one direction with regard to a given point, be equal to the sum of the moments of those that act in the opposite direction with regard to the same point, that point will be in the resultant of the forces; for it can be proved as above, that if this point were fixed there would be equilibrium.

Cor. 2. The moment of the resultant of any number of forces in the same plane, with regard to any point in that plane, is equal to the excess of the sum of the moments of those that act in one direction with regard to that point, over the sum of the moments of those that act in the opposite direction; for if a force were applied equal and opposite to this resultant, there would be equilibrium, and therefore the sums of the moments would be equal to one another.

Prop. 12. The resultant of two parallel forces is in a right line parallel to them, which cuts the line joining their points of application in the inverse ratio of the forces, if they act in the same direction; and, if they act in opposite directions, cuts the same line produced beyond the point of application of the greater force, so that the whole produced line is to the produced part in the ratio of the forces.





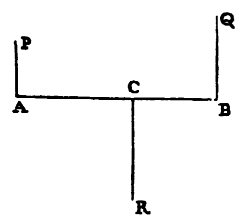
For draw any two right lines, AB, CD, perpendicular to the directions of the given forces AP, BQ, and cut them at F and G in the inverse ratio of the forces, then the moment AP multiplied by AF, is equal to BQ multiplied by BF, and their directions are opposite with regard to F,

therefore if the point F were fixed, there would be equilibrium, and therefore the resultant of the forces AP, BQ, passes through this point F (Ax. 6); in like manner it can be proved, that this resultant passes through the point G, therefore it is in the direction FG, but it is evident that this line is parallel to AP and BQ, and that it cuts every line joining these lines in the inverse ratio of the forces; and since the moments with regard to F are in opposite directions, it is also evident, that this point lies between AP and BQ if they act in the same direction, and that it lies in the production of AB, if they act in opposite directions.

Cor. If the forces AP and BQ be equal and in opposite directions, it is evidently impossible to find a point F in the production of AB in either direction, such, that AF shall be to BF in the inverse ratio of the forces, therefore in this case there is no single resultant.

Prop. 13. The resultant of two parallel forces is equal to their sum if they act in the same direction, and equal to their difference if they act in opposite directions.

1. Let AP, BQ, be two parallel forces acting in the same direction, and cut AB so that AC is to CB as BQ to AP, then the resultant of these two forces will be in the right line drawn through C, parallel to AP and BQ (Prop. 12); let CR be equal and opposite to this resultant,

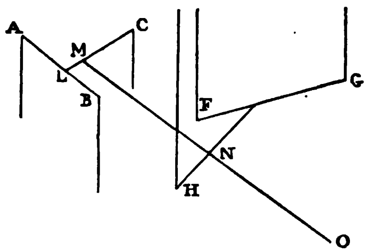


therefore the three forces AP, BQ, CR, are in equilibrio, and therefore BQ is directly opposite to the resultant of AP and CR, therefore AP is to CR as BC to AB (Prop. 12); but since AP is to QB as BC to AC, AP is to the sum of AP and BQ as BC to AB, therefore AP is to the sum of AP and BQ as AP to CR, and CR is equal to the sum of AP and BQ.

2. Let AP, CR, be two parallel forces acting in opposite directions, and let BQ be equal and opposite to their resultant; then since it must act in the same direction as one of

them AP, and since the three forces are in equilibrio, the other force CR is equal to the sum of AP and BQ (Case 1), therefore BQ is the difference between AP and CR.

Prop. 14. To find the resultant of any number of parallel forces A, B, C, F, G, H, &c.

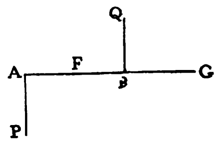


Join A, B, the points of application of two forces which act in one direction, and cut AB in L, so that AL is to BL as B to A, then the resultant of A and B is a force at L equal to their sum (Prop. 12 and 13); join CL and cut it at M, so that CM is to ML as the sum of A and B to C, and the resultant of the three forces A, B, C, is a force at M equal to their sum; for this force is the resultant of C and of another force at L, which has been already proved to be the resultant of A and B; in like manner, if there was a fourth force acting in the same direction, the resultant of it, and the force at M, might be found; and so on for any greater number of forces. And in the same manner the resultant of the forces F, G, H, which act in the opposite direction, may be found; and if this resultant be equal to the resultant of A, B, C, and in the same right line, there will be equilibrium; if the resultants be equal, and not in the same right line, they will have no single resultant (Cor. Prop. 12); and if they be unequal, let the resultant of F, G, H, be the greater, and let N be its point of application; join MN and produce it to O, so that MO is to NO as the sum of F, G, and H to the sum of A, B, and C, then O will be the point of application of the resultant of all the forces (Prop. 12), and it will be equal to the difference between the sum of A, B, and C, and the sum of F, G, and H (Prop. 13).

Note. The point O is called the centre of the system of

parallel forces, and it is evident that its position is independent of the direction of the forces, provided only that A, B, and C act in one direction, and F, G, H in the opposite direction.

Prop. 15. The moment of a pair of equal parallel forces in opposite directions with regard to any point F in the plane of the forces, is equal to their moment with regard to any other point G in the same plane.



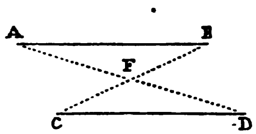
For if the point F lie between the directions of the forces, their moments will be in the same direction with regard to it, and therefore the moment of the pair will be equal to the sum of their separate moments, that is equal to one of the equal forces AP, multiplied by AB, the sum of the distances AF and FB; and if the point G lie on the other side of the line BQ, the moments will be in opposite directions, and therefore the moment of the pair will in this case be equal to their difference, that is, equal to one of the equal forces AP, multiplied by AB, the difference of the distances AG and BG; therefore the moment with regard to G is equal to the moment with regard to F.

Prop. 16. Any number of pairs of equal and opposite parallel forces, in the same plane, will make equilibrium if the sum of the moments of those that act in one direction with regard to any point F in that plane, be equal to the sum of the moments of those that act in the opposite direction with regard to the same point.

For, if they do not make equilibrium, they must have a single resultant passing through the point F (Cor. Prop. 11); but since their moments with regard to any other point G in the same plane, are respectively equal to their moments with regard to F (Prop. 15), it may be proved in like manner, that this single resultant passes through G, and through every other point in the plane, which is absurd; therefore there is equilibrium.

Cor. Any pair of such forces is equivalent to any other pair of equal moment in the same plane.

Prop. 17. Two pairs of equal and opposite parallel forces in parallel planes will make equilibrium, if their moments be equal and opposite.



1. Let the forces of one pair be equal and parallel to the forces of the other pair, then their distances must be also equal; let the forces be supposed perpendicular to the plane of the figure at the points A and B, C and D, and let one pair act upwards at A, and downwards at B, then the other pair being of opposite moment, must act downwards at C, and upwards at D; but since the planes of the pairs are parallel, the right lines AB and CD, in which these planes intersect the plane of the figure, are parallel, and they are also equal, therefore the right lines AD and BC bisect one another at F; and since the forces are all equal to one another, the resultant of the two upward forces at A and D, is an upward force at F, equal to their sum; and the resultant of the two downward forces at B and C is a downward force at F equal to their sum, therefore, these resultants being equal and directly opposite, there is equilibrium.

2. Let the forces of one pair be either unequal or not parallel to those of the other pair, and suppose a third pair of equal moment in the second plane, whose forces are equal and parallel to those of the first, this pair is equivalent to the given pair in the second plane (Cor. Prop. 16); but it makes equilibrium with the pair in the first plane (Case 1); therefore, the given pair to which it is equivalent, will also make equilibrium with the first pair.

Cor. 1. A pair of such forces in any plane is equivalent to a pair of equal moment in any parallel plane.

Cor. 2. Any number of pairs of equal and opposite parallel forces, in parallel planes, will make equilibrium if the sum of

the moments of those that act in one direction be equal to the sum of the moments of those that act in the opposite direction.

Prop. 18. Two pairs of equal and opposite parallel forces in intersecting planes do not make equilibrium.

For each of them is equivalent to a pair of equal moment, one of whose forces is parallel to, and the other coincides with, the intersection of the two planes; therefore the forces are equivalent to three parallel forces which are not in the same plane, and therefore do not make equilibrium (Prop. 12).

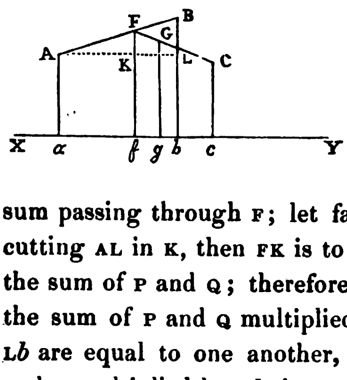
Cor. It is evident, that the two forces which coincide with the intersection of the planes, have a resultant equal to their sum or difference, according as they act in the same or opposite directions; and the other two forces have also a resultant equal and parallel to this in the opposite direction; therefore the two given pairs have a resultant pair of the same nature. If the two forces which coincide with the intersection of the planes be equal and opposite, they will counteract one another, and the remaining two will be themselves the resultant pair.

Prop. 19. Any number of forces acting in parallel planes, on a rigid body containing a fixed axis perpendicular to these planes, will make equilibrium, if the sum of the moments of those that act in one direction with regard to this axis, be equal to the sum of the moments of those that act in the opposite direction.

For suppose that at the point where the axis intersects each plane, a force is applied equal and parallel to the force which acts in that plane, and in the opposite direction; there will thus be formed as many pairs of equal and opposite parallel forces, as there were given forces, and the moment of each pair will be equal to the moment with regard to the axis of the corresponding given force (each being equal to that force multiplied by its perpendicular distance from the axis), therefore the sum of all the moments of these pairs of

parallel forces in one direction, is equal to the sum of those in the opposite direction, and they are in parallel planes, therefore there is equilibrium (Cor. 2, Prop. 17); but the additional forces, being applied to the fixed axis, are equilibrated by its resistance, and therefore may be suppressed without disturbing the equilibrium; therefore the given forces must also make equilibrium.

Prop. 20(*a*). If any number of parallel forces, P, Q, R, &c., act in the same direction on given points, A, B, C, their resultant will pass through a point whose distance from any plane XY, multiplied by the sum of all the forces, is equal to the sum of the products of each force, multiplied by the distance of its point of application from the same plane.



Let A*a* and B*b* be perpendicular to the plane XY, join *ab*, and draw AL parallel to *ab*, and cut AB in F, so that AF is to FB as Q to P, then the resultant of the forces P and Q, will be a force equal to their sum passing through F; let fall F*f* perpendicular on *ab*, and cutting AL in K, then FK is to BL as AF to AB, that is, as Q to the sum of P and Q; therefore Q multiplied by BL is equal to the sum of P and Q multiplied by FK; but since A*a*, K*f*, and L*b* are equal to one another, the sum of P multiplied by A*a*, and Q multiplied by L*b*, is equal to the sum of P and Q multi-

(*a*) The product of a force multiplied by the perpendicular distance of its point of application from any plane, is called the moment of the force with regard to that plane, hence this proposition may be expressed thus: "The sum of the moments of any number of parallel forces with regard to any plane, is equal to the moment of their resultant with regard to the same plane;" it is easy to see, that if any of the forces acted in the opposite direction, or if their points of application were on the opposite side of the plane, they should be treated as negative quantities; and that the moments are to be considered positive or negative, according as they are the products of quantities of like or of unlike signs.

plied by $\text{K}f$, and if these equals be added to the former equals, their sums will be equal, that is, the sum of P multiplied by $\text{A}a$, and Q multiplied by $\text{B}b$, equal to the sum of P and Q multiplied by $\text{F}f$; in like manner joining FC and cutting it in G, so that FG is to GC as R to the sum of P and Q, the resultant of the three forces P, Q, and R will pass through the point G; and it may be proved as before, that the sum of P and Q, multiplied by $\text{F}f$, together with R multiplied by $\text{C}c$, is equal to the sum of P, Q, and R multiplied by $\text{G}g$; but the sum of P and Q multiplied by $\text{F}f$, has been already proved equal to the sum of P multiplied by $\text{A}a$, and Q multiplied by $\text{B}b$, therefore the sum of P multiplied by $\text{A}a$, Q multiplied by $\text{B}b$, and R multiplied by $\text{C}c$, is equal to the sum of P, Q, and R multiplied by $\text{G}g$; and in like manner it may be proved for any greater number of forces.

NOTE TO CHAPTER I.

ANY number of forces $\text{P}, \text{P}', \text{P}''$, acting on different points of a rigid body, may, by Prop. 7, be resolved each into three, parallel to three rectangular axes of coordinates OX, OY, OZ, and thus will be formed three systems of parallel forces, whose conditions of equilibrium are easily determined.

Let the angles which the directions of the given forces make with right lines parallel to the three axes of coordinates be α, β, γ; α', β', γ'; $\alpha'', \beta'', \gamma''$, &c.; and the coordinates of their points of application x, y, z; x', y', z'; x'', y'', z'', &c.; then the system of forces parallel to OX will be $\text{P} \cos \alpha$, $\text{P}' \cos \alpha'$, $\text{P}'' \cos \alpha''$, &c.; those parallel to OY will be $\text{P} \cos \beta$, $\text{P}' \cos \beta'$, $\text{P}'' \cos \beta''$; and those parallel to OZ, $\text{P} \cos \gamma$, $\text{P}' \cos \gamma'$, $\text{P}'' \cos \gamma''$, &c. (Cor. 2, Prop. 7); now, if there be equilibrium among these forces, it will not be disturbed by supposing the axis OZ to be fixed, on which supposition the forces $\text{P} \cos \gamma$, $\text{P}' \cos \gamma'$, &c., being each in the same plane with this axis, will be equilibrated by its resistance (Cor. 1, Prop. 6), and the forces $\text{P} \cos \alpha$, $\text{P}' \cos \alpha'$, &c., being in planes perpendicular to the fixed axis, will make equilibrium when the sum of their moments with regard to this axis is equal to cypher (Prop. 19); the moments of those which act in one direction being positive, and those in the opposite direction negative; but the moments of the forces parallel to OX are $\text{P} \cos \alpha \,.\, y$, $\text{P}' \cos \alpha' \,.\, y'$, $\text{P}'' \cos \alpha'' \,.\, y''$, &c.; and the moments of those parallel to OY are $\text{P} \cos \beta \,.\, x$, $\text{P}' \cos \beta' \,.\, x'$, $\text{P}'' \cos \beta'' \,.\, x''$, &c.; moreover, if both the forces and their coordinates be positive, it is evi-

dent that the moment of those parallel to OX is in the opposite direction to the moment of those parallel to OY; therefore, if one of these is accounted positive, the other must be negative, and the sum of their moments is $(P\cos\alpha\,.\,y + P'\cos\alpha'.\,y' + P''\cos\alpha''.\,y'' + \&c.) - (P\cos\beta\,.\,x + P'\cos\beta'.\,x' + P''\cos\beta''.\,x'' + \&c.) = N$; and the condition of equilibrium when OZ is fixed is $N = 0$; in like manner, if $M = (P\cos\gamma\,.\,x + P'\cos\gamma'.\,x' + \&c.) - (P\cos\alpha\,.\,z + P'\cos\alpha'.\,z' + P''\cos\alpha''.\,z'' + \&c.)$; and $L = (P\cos\beta\,.\,z + P'\cos\beta'.\,z' + P''\cos\beta''.\,z'' + \&c.) - (P\cos\gamma\,.\,y + P'\cos\gamma'.\,y' + P''\cos\gamma''.\,y'' + \&c.)$; the condition of equilibrium will be, when OY is fixed, $M = 0$, and when OX is fixed, $L = 0$; consequently, if the three equations be satisfied

$$L = 0, \quad M = 0, \quad N = 0, \tag{1}$$

equilibrium will be produced by fixing any one of the three axes, and therefore the given forces must be equivalent to forces which act in the same plane with each of the three axes, that is to say, which pass through the origin of coordinates where these three axes intersect; therefore they will either make equilibrium, or have a single resultant passing through this point. Now suppose that the axes of coordinates were transferred in parallel directions to a point whose coordinates are a, b, c, and let the moment of the forces with regard to the three axes in their new position, be L', M', N', then

$$L' = L - (P\cos\beta + P'\cos\beta' + P''\cos\beta'' + \&c.)c + (P\cos\gamma + P'\cos\gamma' + P''\cos\gamma'' + \&c.)b,$$
$$M' = M - (P\cos\gamma + P'\cos\gamma' + P''\cos\gamma'' + \&c.)a + (P\cos\alpha + P'\cos\alpha' + P''\cos\alpha'' + \&c.)c,$$
$$N' = N - (P\cos\alpha + P'\cos\alpha' + P''\cos\alpha'' + \&c.)b + (P\cos\beta + P'\cos\beta' + P''\cos\beta'' + \&c.)a.$$

And if these quantities are also equal to cypher, the resultant (if there be one) must pass through the origin in its new position; also let

$$P\cos\alpha + P'\cos\alpha' + \&c. = X,\quad P\cos\beta + P'\cos\beta' + \&c. = Y,\quad P\cos\gamma + P'\cos\gamma' + \&c. = Z;$$

and this condition will be thus expressed, $L' = L - Yc + Zb = 0$, $M' = M - Za + Xc = 0$, $N' = N - Xb + Ya = 0$; and subtracting equations (1) from these there results, $Xb - Ya = 0$, $Xc - Za = 0$, $Yc - Zb = 0$; the latter of these equations is contained in the two former, and they are in fact the equations of the resultant, a, b, c, being the coordinates of any point of this resultant. If the given forces make equilibrium, the equations $Xb - Ya = 0$, $Xc - Za = 0$, $Yc - Zb = 0$, must be satisfied by all possible values of a, b, and c, which can only be the case when

$$X = 0, \quad Y = 0, \quad Z = 0, \tag{2}$$

the six equations therefore, $X = 0$, $Y = 0$, $Z = 0$, $L = 0$, $M = 0$, $N = 0$, are necessary and sufficient for equilibrium.

If these conditions of equilibrium be not satisfied, let it be proposed to determine whether the forces have a single resultant.

If it be possible, let there be a single resultant R, applied at a point whose coordinates are x, y, z, and making angles A, B, Γ, with right lines parallel to OX, OY, OZ, then if a force be applied equal and opposite to R, there will be equilibrium; and therefore the conditions expressed by equations (1) and (2) must be satisfied, that is to say, $X - R\cos A = 0$, $Y - R\cos B = 0$, $Z - R\cos\Gamma = 0$, $L - R(z\cos B - y\cos\Gamma) = 0$, $M - R(x\cos\Gamma - z\cos A) = 0$, $N - R(y\cos A - x\cos B) = 0$; also $\cos^2 A + \cos^2 B + \cos^2\Gamma = 1$, from which the seven unknown quantities R, x, y, z, A, B, Γ, are to be determined: the first three equations combined with the last, give $R = \sqrt{(X^2 + Y^2 + Z^2)}$, $\cos A = \frac{X}{R}$, $\cos B = \frac{Y}{R}$, $\cos\Gamma = \frac{Z}{R}$, and substituting these values in the three remaining equations, there results $L = Yz - Zy$, $M = Zx - Xz$, $N = Xy - Yx$, and multiplying these equations respectively by X, Y, Z, and adding them, we obtain

$$XL + YM + ZN = 0, \qquad (3)$$

an equation which contains only known quantities, and expresses the condition necessary to be satisfied, in order that there may be a single resultant; when this equation is satisfied, it is evident that the equations $Yz - Zy = L$, $Zx - Xz = M$, $Xy - Yx = N$, are equivalent to only two independent equations, and that they represent the projections of the direction of the resultant, on the three coordinate planes; if $X = 0$, $Y = 0$, $Z = 0$, the coordinates x, y, z, will become infinite, and the resultant $R = 0$; in fact the quantities X, Y, Z are the sums of the forces parallel to the three axes, and are therefore in general equal to their resultants, and if they become equal to cypher, the forces either make equilibrium, or are reduced to a pair of equal and opposite parallel forces, whose resultant, expressed analytically, is equal to cypher and at an infinite distance.

Finally, if the condition $XL + YM + ZN = 0$ does not hold, let $XL + YM + ZN = LX'$, and suppose a force $-X'$ to act in the direction OX, then since the coordinates of its point of application are each $= 0$, the introduction of this force makes no change in the above values of L, M, N, and since, by hypothesis, they were different from cypher, this force will not produce equilibrium; but since $L(X - X') + MY + NZ = 0$, there will be a single resultant, whose equations are $Yz - Zy = L$, $Zx - (X - X')z = M$, $(X - X')y - Yx = N$, and its magnitude $R = \sqrt{\{(X - X')^2 + Y^2 + Z^2\}}$, and the given forces are equivalent to this resultant combined with one equal and opposite to $-X'$; it is evident that the direction of this force is perfectly arbitrary, but it is impossible to combine the two forces X' and R into a single resultant. Hence it appears that, in every case, the given forces either make equilibrium, or are equivalent to one, or at the most to two resultants.

Hence it may be proved, that two forces which are not in the same plane cannot have a single resultant; for let the shortest distance z between the

forces, coincide with the axis OZ, the origin O being on the direction of the force A, and the axis OX parallel to the other force B, OY being at right angles to OX and OZ; then since OX is parallel to B, it does not coincide with A, let them make an angle a, then resolving this force, and combining it with the other force B, we will have $X = B + A \cos a$, $Y = A \sin a$, $Z = 0$, $L = 0$, $M = - Bz$, $N = 0$; therefore, $XL + YM + ZN = - ABz \sin a$, which being different from 0, the forces have not a single resultant.

If all the forces act on the same point, it is evident from Prop. 8, that X, Y, Z are the components of their resultant parallel to the three axes, and therefore in this case it is necessary, and sufficient for equilibrium, that $X = 0$, $Y = 0$, $Z = 0$.

CHAPTER II.

OF THE CENTRE OF GRAVITY.

Gravity, at any given place on the earth's surface, is a force acting equally on all bodies which contain equal quantities of matter; this is proved by experiments on the motion which it produces, as will be hereafter shewn. Its direction is perpendicular to the horizon of the place, and therefore varies with the position of the body on the earth's surface; but within the limits of a body of moderate dimensions, it is, *quam prox.*, a system of equal and parallel forces acting in the same direction on all the equal particles of matter of which the body is composed, and therefore admits of a centre, that is, a point through which the resultant of these forces must pass, whatever be their direction (Chap. I., Prop. 14); this point is called the centre of gravity. If the body be homogeneous, these equal particles of matter are of equal magnitude; but if it be not of equal density in all parts, the number of equal particles contained in a given space, is proportional to the density. The weight of a body is the resultant of all these equal and parallel forces, and is therefore equal to their sum; consequently, the weight of a body is proportional to the number of equal particles of matter which it contains, and is therefore a true measure of the quantity of matter in the body, provided that it be of moderate dimensions, and situated at a given place on the earth's surface. The following propositions are equally true of any system of equal parallel forces, and may therefore be applied in this sense to any bodies whatever.

Prop. 1. Every body has a centre of gravity.

For if it be divided into an indefinite number of infinitely small particles, these particles are the points of application of a

system of parallel forces equal to their weights, therefore the centre of these forces may be found (Chap. I., Prop. 14) by joining two of the particles, and cutting the joining line in the inverse ratio of their weights, and again joining this point of section with a third particle, and cutting the joining line in the inverse ratio of the sum of the first two particles to the third, and so on to the last particle of the body. The point thus found will be the centre of gravity.

Cor. If a body be divided into parts whose centres of gravity are in one right line or plane, the centre of gravity of the whole body is in the same right line or plane.

Prop. 2. No body can have more than one centre of gravity.

For if it be possible, let a body have two centres of gravity, and place it so, that the line joining them shall be horizontal, then since the resultant of the weights of the several particles of the body passes through each of these centres, its direction must be the right line joining them, that is to say, a horizontal line, which is absurd; therefore the body cannot have two centres of gravity.

Prop. 3. If a rigid body contain a fixed point in the vertical passing through its centre of gravity, it will be in equilibrio; and if it be kept in equilibrio by the resistance of a single point, that point must be in the vertical which passes through its centre of gravity.

For since the resultant of the weights of all the particles of the body is a vertical force passing through the centre of gravity, it will be balanced by the resistance of any fixed point in that right line; and it will not be balanced by a fixed point which is not in the line of its direction (Ax. 6).

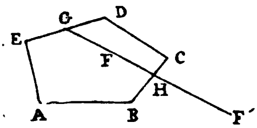
Cor. 1. Hence if the centre of gravity of a body be fixed, the body will balance itself in any position.

Cor. 2. Hence also the centre of gravity of a rigid body may be found easily by suspending the body from two points successively, and marking on the body the vertical line

through the point of suspension in each position, each of these lines must pass through the centre of gravity, therefore their intersection is the point required.

Cor. 3. If a body be placed on a horizontal plane, it will stand or fall according as the vertical which passes through its centre of gravity meets the horizontal plane within or without the figure contained by right lines joining the extreme points A, B, C, D, E, of the base of the body.

For let F be the point where the vertical meets the horizontal plane, and draw the right line GH, cutting two sides of the figure in G and H, then if the point F be between G and H, the weight of the body which acts in the vertical through F, may be resolved into two at G and H, and each of these again may be resolved into two at D and E, B and C, and being at these points perpendicular to the horizontal plane, they will be equilibrated by its resistance; but if the point F′, where the vertical meets the horizontal plane, be in the production of GH, its components at G and H will act in opposite directions, and therefore one will be directed upwards from the horizontal plane, and will not be equilibrated.



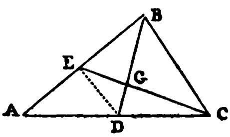
Prop. 4. The centre of gravity of an homogeneous right line, is its point of bisection.

For the line being homogeneous is composed of a number of equal particles placed at equal distances on each side of the point of bisection, and the centre of gravity of each pair of these particles will therefore be at the point of bisection, and therefore the centre of gravity of the whole line is at the same point.

N. B.—In this and all the following propositions, the figures are supposed to be formed of homogeneous matter, and of uniform thickness.

Prop. 5. To find the centre of gravity of the perimeter of a rectilinear figure.

Bisect all the sides of the figure, and find the centre of a system of parallel forces proportional to the sides, and applied at their points of bisection (Chap. I., Prop. 14), this will be the centre of gravity required. For the sides being homogeneous, the weight of each is a force proportional to its length, and applied at its point of bisection (Prop. 4); therefore the resultant of all these weights will be a force equal to their sum, and applied at the centre of these forces.



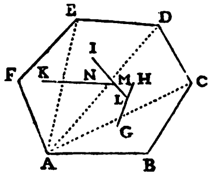
Prop. 6. To find the centre of gravity of a triangle ABC.

Bisect any two sides AB and AC in E and D, and join BD, CE, the point G where these lines intersect, is the centre of gravity.

For the triangle may be divided by a series of lines parallel to AC, and indefinitely close to one another, into a number of homogeneous right lines of uniform thickness, and the centre of gravity of each of these will be its point of bisection; but the points of bisection of all these lines are in the right line BD, therefore the centre of gravity of the whole triangle is in the same right line (Cor. Prop. 1); in like manner it may be shewn that the centre of gravity is in the line CE, therefore it is in G the intersection of these lines.

Cor. 1. The point G is one of the points of trisection of the line BD, for join DE, and since AD is to DC as AE to EB, the line DE is parallel to BC, therefore DG is to GB as DE to BC; but DE is one-half of BC, therefore DG is one-half of BG.

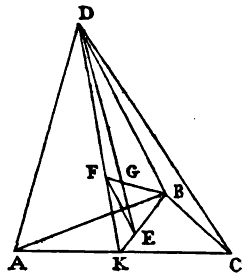
Cor. 2. The right lines drawn from the angles of a triangle to bisect the opposite sides, all pass through the same point, for they all pass through the centre of gravity.



Prop. 7. To find the centre of gravity of any rectilinear figure.

Resolve the figure into triangles, and find the centres of gravity G, H, I, K of the triangles, join GH and cut it at L,

in the inverse ratio of the triangles ABC, ACD, join LI, and cut it at M in the inverse ratio of the triangle ADE to the quadrilateral ABCD, and so on, as in Prop. 1. The point thus found will be the centre of gravity, for the forces acting at each of the points G, H, I, K are the weights of the triangles, and are therefore proportional to their areas, therefore their centre is to be found as in Prop. 1.



Prop. 8. To find the centre of gravity of a triangular pyramid ABCD.

Find the centre of gravity E of the face ABC, and join DE, and find the centre of gravity F of the face ACD, and join BF, these lines will intersect in G, the centre of gravity of the pyramid.

For the pyramid may be divided into a number of similar triangles, by planes parallel to ABC, and the right line DE will evidently pass through the centre of gravity of each of these triangles, therefore the centre of gravity of the entire pyramid is in this line; and in like manner it may be proved, that it is in the right line BF, therefore these lines intersect at the centre of gravity.

Cor. 1. The four right lines drawn from the vertices A, B, C, D to the centres of gravity of the opposite faces intersect in the same point.

Cor. 2. The centre of gravity of a triangular pyramid is at a distance from the base equal to one-fourth part of the altitude.

For let K be the point of bisection of the edge AC, and join BK, DK, and take parts EK, FK respectively equal to one-third of BK and DK, the points E and F will be the centres of gravity of the faces ABC, ADC (Cor. 1, Prop. 6), join BF and DE, and their intersection G is the centre of gravity of the pyramid (Prop. 8); join FE, and since FK, EK are the third parts of the sides of the triangle DKB, EF will be one-third of its

base BD, and parallel to it; therefore EG is one-third of DG, and therefore it is one-fourth of DE; and if perpendiculars be drawn from D and G to the base ABC, the perpendicular from G will also be one-fourth of the perpendicular from D.

Prop. 9. To find the centre of gravity of a pyramid whose base is a rectilinear figure of any form.

Find the centre of gravity of the base (Prop. 7), and draw a right line from it to the vertex; cut off a fourth part of this line from the base, and the point thus found will be the centre of gravity of the pyramid.

For the pyramid may be divided into a number of similar rectilinear figures, by planes parallel to the base, and it is evident that their centres of gravity will all lie on the right line drawn from the vertex of the pyramid to the centre of gravity of the base, therefore the centre of gravity of the whole pyramid will lie in the same right line; but since the pyramid may also be resolved into triangular pyramids having the same vertex, and the triangles, of which the given base is composed, for bases; and since the centre of gravity of each of these pyramids is at a distance from the base equal to one-fourth of the altitude of the pyramid, their centres of gravity are therefore all in the same plane parallel to the base, and therefore the centre of gravity of the whole pyramid is in this plane; but this plane cuts off one-fourth part of the right line drawn from the vertex to the centre of gravity of the base, and the point of intersection is the centre of gravity of the pyramid.

Cor. The centre of gravity of a cone may be found in the same way, if the centre of gravity of its base be known; for the cone may be considered as a pyramid whose base has an infinite number of sides, and therefore its centre of gravity is found by cutting off one-fourth part of the line drawn from the centre of gravity of the base to the vertex; thus, if the base of the cone be a circle, or an ellipse, the centre of gravity of the cone is in the line drawn from the centre of the

base to the vertex, and at a distance from the base equal to one-fourth part of the altitude of the cone.

Prop. 10. If a body be divided into any number of parts, the sum of the products of the weight of each part multiplied by the distance of its centre of gravity from any plane, is equal to the product of the weight of the whole body multiplied by the distance of its centre of gravity from the same plane.

For the weights of the different parts are a number of parallel forces acting in the same direction on their centres of gravity, and the centre of gravity of the whole body is the point of application of their resultant; therefore, this proposition is a case of Prop. 20, Chap. I.

Note to Chapter II.

The method of finding the centre of gravity of a line, surface, or solid, analytically, involves the principles of the integral calculus; the property of the centre of gravity which is most conveniently used for this purpose, is that which was proved in Prop. 10, by which the three coordinates of this point are immediately found; thus if it be proposed to find the centre of gravity of a curve line of uniform thickness, whether plane or of double curvature, each coordinate of this point multiplied by the entire length of the curve is, by the proposition referred to, equal to the sum of each element of the curve multiplied by its coordinate; thus if s be the entire length of the curve, and x, y, z the coordinates of the indefinitely small portion of this length ds, and $\bar{x}$, $\bar{y}$, $\bar{z}$, the coordinates of the centre of gravity, we have $\bar{x}s =$ the sum $xds +$ the corresponding product for each of the small portions of which the curve is composed, that is, adopting the notation of the integral calculus, $= \int xds$, and $\bar{x} = \dfrac{\int xds}{s}$, in like manner $\bar{y} = \dfrac{\int yds}{s}$, $\bar{z} = \dfrac{\int zds}{s}$, where the quantities s and ds are to be found from the equations of the curve; if these equations be referred to rectangular coordinates, it is evident that the small arc ds is the diagonal of a rectangular parallelopiped, of which the differences of the coordinates of the two extremities of this arc are the sides, and calling these differences dx, dy, dz; $ds^2 = dx^2 + dy^2 + dz^2$, and $ds = dx \sqrt{\left(1 + \dfrac{dy^2}{dx^2} + \dfrac{dz^2}{dx^2}\right)}$, or calling the quantities $\dfrac{dy}{dx}$ and $\dfrac{dz}{dx}$ (which are deter-

mined by means of the equations of the curve) p and q, $ds = dx\sqrt{(1+p^2+q^2)}$, and as s is the sum of all these small portions, adopting the same notation as before, $s = \int\sqrt{(1+p^2+q^2)}\,dx$, and $\bar{x} = \frac{\int\sqrt{(1+p^2+q^2)}\,xdx}{\int\sqrt{(1+p^2+q^2)}\,dx}$, $\bar{y} = \frac{\int\sqrt{(1+p^2+q^2)}\,ydx}{\int\sqrt{(1+p^2+q^2)}\,dx}$, $\bar{z} = \frac{\int\sqrt{(1+p^2+q^2)}\,zdx}{\int\sqrt{(1+p^2+q^2)}\,dx}$, in which equations it is only necessary, previous to the integration, to substitute for y, z, p and q, their values derived from the equations of the curve.

For example, let it be proposed to find the centre of gravity of an arc of a circle, assume the centre as origin, and the plane of the circle as the plane of xy, the axis of x passing through an extremity of the given arc; then the equations of the circle will be $z = 0$, $x^2 + y^2 = r^2$, and by differentiation $q \left(= \frac{dz}{dx}\right) = 0$, $xdx + ydy = 0$, therefore $p \left(= \frac{dy}{dx}\right) = -\frac{x}{y}$, $\sqrt{(1 + p^2 + q^2)} = \frac{-\sqrt{(x^2+y^2)}}{y} = -\frac{r}{y}$, and substituting these values in the expressions for the coordinates of the centre of gravity $\bar{x} = -\frac{\int\frac{r}{y}\,xdx}{s}$, $\bar{y} = -\frac{\int rdx}{s}$, $\bar{z} = 0$; let y', x', be the coordinates of one extremity of the arc, the coordinates of the other extremity being 0, r, then, taking the integrals between these limits, $\bar{y} = \frac{r(r-x')}{s}$, $\bar{x} = -\frac{\int\frac{r}{y}\,xdx}{s} = +\frac{\int\frac{r}{y}\,ydy}{s} = +\frac{ry'}{s}$; but if $s = r\theta$, then $y' = r\sin\theta$, $x' = r\cos\theta$, and these equations become $\bar{y} = r\,\frac{\text{versin}\,\theta}{\theta}$, $\bar{x} = r\,\frac{\sin\theta}{\theta}$.

Next let it be proposed to find the centre of gravity of a surface, and for simplicity let it be a plane surface bounded by one or more lines, whose equations referred to rectangular axes OX, OY are known, and let this surface be divided into indefinitely small parts, by right lines parallel to OY, and at a distance dx from one another, the area of one of these small surfaces will be $(y' - y'')\,dx$, where y', y'' are the ordinates of the two extremities of one of these parallel lines, and are determined by the equations of the lines by which the surface is bounded. Let $y = fx$ be the equation of one of these lines, and $y = \text{F}x$ that of the other, then the area will be $(\text{F}x - fx)\,dx$, and the distance of its centre of gravity from OY is x, therefore, by the property of the centre of gravity, $\bar{x} \times \int(\text{F}x - fx)\,dx = \int(\text{F}x - fx)\,xdx$, or $\bar{x} = \frac{\int(\text{F}x - fx)\,xdx}{\int(\text{F}x - fx)\,dx}$, the integrals being taken between the extreme limits

of x; the distance of the centre of gravity of one of the elementary portions from OY is $\frac{y'+y''}{2}$, or $\frac{(Fx+fx)}{2}$, and the product of this by the area of the same element is $\frac{(Fx)^2-(fx)^2}{2}\,dx$, therefore $\bar{y}=\frac{\int((Fx)^2-(fx)^2)\,dx}{2\int(Fx-fx)\,dx}$.

Thus let it be proposed to find the centre of gravity of the area contained by the arc of a parabola, its axis, and an ordinate. The equations of these lines referred to the axis and a tangent at its vertex, are $y=\sqrt{(px)}$, $y=0$. $x=a$; substituting the above values of y for Fx and fx, the coordinates of the centre of gravity become

$$\bar{x}=\frac{\int\sqrt{(px)}\;xdx}{\int\sqrt{(px)}\;dx}=\frac{\sqrt{p}\,.\,\frac{2}{5}x^{\frac{5}{2}}+C}{\sqrt{p}.\frac{2}{3}x^{\frac{3}{2}}+C'}.$$

$$\bar{y}=\frac{\int px\;dx}{2\int\sqrt{(px)}\;dx}=\frac{\frac{1}{2}p\,x^2+C''}{\sqrt{(p)}\cdot\frac{4}{3}\,x^{\frac{3}{2}}+C'''},$$

and taking these between the limits $x=0$ and $x=a$, we have finally $\bar{x}=\frac{3}{5}\,a$ $\bar{y}=\frac{3}{8}\sqrt{(pa)}$.

For another example, let it be proposed to find the centre of gravity of a semicircle bounded by the lines $y^2+x^2=r^2$, $y=0$, in this case therefore

$$\bar{x}=\frac{\int\sqrt{(r^2-x^2)}\;xdx}{\int\sqrt{(r^2-x^2)}\;dx},\;\bar{y}=\frac{\frac{1}{2}\int(r^2-x^2)\;dx}{\int\sqrt{(r^2-x^2)}\;dx};$$

the denominator of each of these is evidently the area of the semicircle $\frac{\pi r^2}{2}$, therefore we have, taking the integrals of the numerators between the extreme limits $+r, -r$,

$$\bar{x}=0,$$
$$\bar{y}=\frac{r^3-\frac{1}{3}r^3}{\pi r^2}-\frac{-r^3+\frac{1}{3}r^3}{\pi r^2}=\frac{4r}{3\pi},$$

that is, the centre of gravity is on the axis of y, at a distance from the centre of the circle equal to $\frac{2}{3}$ of a third proportional to the circumference and diameter.

To find the centre of gravity of a solid, bounded by surfaces whose equations are known, suppose it to be divided into an indefinite number of thin plates perpendicular to the axis OX, let the area of one of these be A, and the thickness of the plate dx, then this plate will be equal to Adx, and the whole solid $\int Adx$, and it follows as before, that $\bar{x}=\frac{\int Axdx}{\int Adx}$ where A is $\int zdy$, in like manner $\bar{y}$ and $\bar{z}$ are found by dividing the solid by planes perpen-

dicular to OY and to OZ, whence we have the following values, $\bar{x} = \frac{\iint zdyxdx}{\iint zdydx}$, $\bar{y} = \frac{\iint zdxydy}{\iint zdxdy}$, $\bar{z} = \frac{\iint ydxzdz}{\iint ydxdz}$; in each of which fractions the denominator is the volume of the solid, and the numerator is found by taking the first integral between the extreme limits of the coordinates in each elementary plane, and the second between the extreme values of the coordinate perpendicular to these planes; thus let it be proposed to find the centre of gravity of a hemisphere, bounded by the surfaces $x^2 + y^2 + z^2 = r^2$, $z = 0$, it is evident that the centre of gravity must be in the axis of z, therefore $\bar{x} = 0$, $\bar{y} = 0$, and it only remains to find the third coordinate $\bar{z}$; now the equation of a plane section perpendicular to OZ is $x^2 + y = r^2 - z^2$ (where z is constant for each section), therefore $\int ydx = \int \sqrt{(r^2 - z^2 - x^2)}\, dx$, which integral is to be taken between the limits $x = +\sqrt{(r^2 - z^2)}$, $x = -\sqrt{(r^2 - z^2)}$, but it is not necessary in the present case to perform the integration, as the result is evidently the area of a circle, whose radius is $\sqrt{(r^2 - z^2)}$, therefore $\int ydx = \pi\, (r^2 - z^2)$ and $\bar{z} = \frac{\int \pi\, (r^2 - z^2)\, zdz}{\int \pi\, (r^2 - z^2)\, dz}$ integrating between the limits $z = 0$, $z = r$, we have $\bar{z} = \frac{3r^4}{8r^3} = \frac{3}{8}r$.

If it was proposed to find the centre of gravity of a segment of a sphere cut off by a plane surface, the same equation will give the answer, by taking the integral between the limits $z = r$, and $z = a$, where a is the distance of this plane from the centre, hence we have $\bar{z} = \frac{6r^4 - 6r^2a^2 - 3r^4 + 3a^4}{12r^3 - 12r^2a - 4r^3 + 4a^3}$ $= \frac{3r^4 - 6r^2a^2 + 3a^4}{8r^3 - 12r^2a + 4a^3} = \frac{3r^3 + 3r^2a - 3ra^2 - 3a^3}{8r^2 - 4ra - 4a^2} = \frac{3}{4}\frac{(r + a)^2}{(2r - a)}$.

The quadratures and cubatures of figures of revolution furnish us with a method of finding the centres of gravity of plane figures, without introducing the notation of the calculus, for it is evident, that if a surface be generated by the revolution of a plane curve about an axis in its plane, the area of this surface will be equal to the sum of the products of each small portion of the generating line, multiplied by the circumference of the circle which it describes, that is, equal to $2\pi \times$ the sum of the products of each of these small portions into its distance from the axis, which by the property of the centre of gravity is equal to $2\pi \times$ the length of the generating line multiplied by the distance of the centre of gravity of this line from the axis, whence if this distance be $\bar{x}$, we have $\bar{x} = \frac{\text{area of surface of revolution}}{2\pi \times \text{length of generating line}}$; thus, to find the distance $\bar{x}$ of the centr of gravity of the circumference of a semicircle from the diameter, we have $\bar{x} = \frac{\text{area of surface of sphere}}{2\pi \,.\, \pi r}$;

but the area of the surface of a sphere is equal to four times the area of a great circle $= 4\,\pi r^2$; whence $x = \frac{4\,\pi r^2}{2\,\pi^2 r} = \frac{2r}{\pi}$.

In like manner the volume of a solid generated by the revolution of a plane surface about an axis in its plane is equal to the sum of the volumes of the cylindrical shells generated by each of the lines parallel to this axis, of which the plane surface is composed, that is, equal to the sum of the products of the thickness of each of these shells, multiplied by its surface; but the surface of a right cylinder is equal to the length of the generating line multiplied by the circumference of its base, therefore the volume of each shell is this circumference multiplied by the area intercepted between two of these parallel lines, and the sum of all these is 2π multiplied by the sum of each of these small areas multiplied respectively by their distances from the fixed axis, that is equal to $2\pi \times$ the whole area of the generating surface multiplied by the distance of its centre of gravity from the fixed axis; therefore finally, this distance is equal to $\frac{\text{volume of the solid of revolution}}{2\pi \times \text{area of generating surface}}$; thus if we take, as before, the area of a semicircle, the distance of its centre of gravity from the diameter is equal to $\frac{\text{the solid content of a sphere}}{2\pi \times \text{area of } \frac{1}{2} \text{ great circle}} = \frac{\frac{4}{3}\pi r^3}{\pi^2 r^2} = \frac{4r}{3\pi}$, the same value as was before obtained. This property of the centre of gravity, discovered by Guldin, may be thus expressed: *If a surface or solid be generated by the revolution of a plane figure about an axis in its own plane, the area of the surface or volume of the solid is equal to the product of the generating line or surface multiplied by the circumference of the circle described by its centre of gravity.* It may evidently be made use of for the quadrature or cubature of figures of revolution, when the centre of gravity of the generating line, or surface, is known; thus, if a circle be made to revolve about an axis in its own plane, and outside of its circumference, it will describe a ring whose solid content is therefore equal to the area of the generating circle multiplied by the circle described by its centre of gravity, that is, if r be the radius of the generating circle, and R the radius of the circle described by its centre, the contents of the ring will be $\pi r^2 . 2\pi R = 2\pi^2 R r^2$; but it is to be observed, that, in this proposition, it is supposed that the whole of the generating line or surface lies at one side of the axis of revolution; for if any part of it lay at the other side, it is evident that the surface or solid generated by it should be subtracted; thus, if a sphere be considered as generated by the revolution of a circle, instead of a semicircle, the part described by one half of the circumference should be subtracted from that described by the other half, and the difference being 0 proves that the centre of gravity of the generating circle is on the axis, as is otherwise evident.

CHAPTER III.

OF THE MECHANICAL POWERS.

A MACHINE(*a*) is defined to be an instrument by means of which a given force is caused to make equilibrium with a resistance to which it is either unequal, or not directly opposed. The given force may be either the muscular force of animals, the elastic force of steam, or any other of those usually employed in machinery, and for distinction is called THE POWER; and the resistance, or that force which impedes the work to be done, and which it is therefore necessary to counterbalance, is technically called THE WEIGHT.

The simple parts of which machines are composed are called mechanical powers; they are usually reduced to six classes, namely, the lever, the axle and wheel, the pulley, the inclined plane, the screw, and the wedge.

THE LEVER

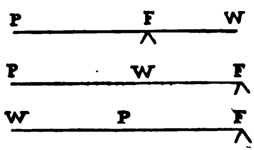
Is a rigid bar, capable of motion about a fixed point or axis, which is called the fulcrum; the power and the weight both act in the plane perpendicular to this axis, that is to say, in the plane in which the lever is capable of moving.

There are three kinds of levers, distinguished by the positions of the points where the power and the weight are applied. In the first kind the power and weight act at opposite sides of the fulcrum, in the second kind the weight is between the

(*a*) The consideration of machines may at first sight appear to belong rather to Dynamics than to Statics, for the practical use of machines is to produce motion, not equilibrium; but since all questions concerning the motions of bodies require first a knowledge of the conditions of their equilibrium, the statical consideration of machines is a necessary introduction to any reasoning concerning their motions.

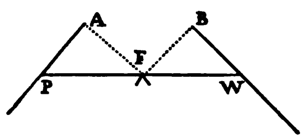
power and fulcrum, and in the third the power is between the weight and fulcrum.

In all the kinds of lever it is evident, that the condition of equilibrium is that the moments of the power and weight, with regard to the fulcrum, should be equal and opposite; see Chap. I., Prop. 10.



Hence if the power P and weight W act perpendicularly to the arms of a lever whose fulcrum is F, P multiplied by PF is equal to W multiplied by WF, that is, the power is to the weight inversely as the arms, P to W as WF to PF; in order that these moments should be opposite, it is necessary that the power and weight should act in the same direction in the first kind of lever, and in opposite directions in the second and third kinds; in the second kind of lever PF is greater than WF, therefore the weight is greater than the power, in the third kind the weight is less than the power; therefore there is said to be a gain of power in the second kind, and a loss in the third; in the first kind there is a gain or loss of power according as the arm to which the power is applied is longer or shorter than that which bears the weight. It is evident, however, that if the lever be used for producing motion instead of equilibrium, rapidity of motion may in some cases be a more important consideration than great force; and in the second kind of lever, the velocity with which the weight is moved is always less than that of the power, in the third kind it is greater; thus, in the lever, when power is gained velocity is always lost: this is not peculiar to the lever, but, as we shall afterwards see, is common to all the mechanical powers. This may account for the fact, that the limbs of animals are composed of levers of the third kind, or, if they are of the first kind, they are always such that there is a loss of power with a corresponding gain of velocity, hich is evi-

dently of much greater importance; this is one of many instances which might be pointed out of the admirable adaptation of animal mechanism to the purposes for which it is designed.



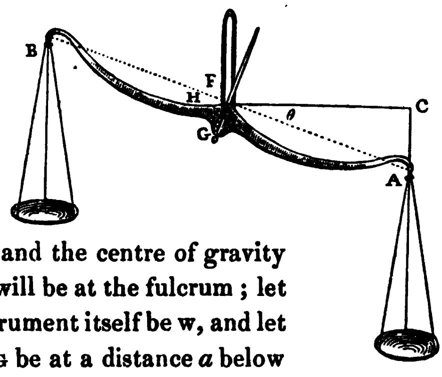
If the power and weight act obliquely to the arms of the lever, their moments with regard to the fulcrum will be found by letting fall the perpendiculars FA, FB on their directions; then the condition of equilibrium is, that P multiplied by FA be equal to W multiplied by FB, or P to W as FB to FA, that is, the power is to the weight *inversely* as the perpendiculars let fall from the fulcrum on their directions. When PFW is a right line, the lever is called a straight lever, and when the arm PF is inclined to PW, it is called a bent lever; it is evident that the conditions of equilibrium are the same in the bent as in the straight lever.

The lever is a convenient instrument for ascertaining the weight of a body by comparing it with another whose weight is known. The weights of the body to be weighed, and of the counterpoise with which it is compared, are two parallel forces acting in the same direction, and may therefore be caused to balance each other by means of a lever of the first kind. Let W be the body to be weighed and P the counterpoise, then, if they do not balance one another, equilibrium may be produced either by changing the counterpoise P, changing its position, changing the place of the fulcrum, or changing the inclination of the arms of the lever to the direction of the forces.

The first of these methods is used in the balance with equal weights, in which the positions of the fulcrum and of the points of suspension of the weight and its counterpoise are fixed, and the balance is so constructed, that it will be horizontal when the weight and its counterpoise are equal. For this purpose it is necessary that the arms of the lever should be equal, and also that the centre of gravity of the instru-

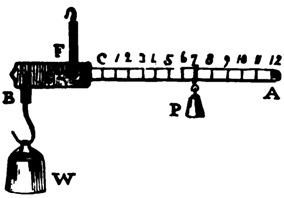
ment itself should be in the perpendicular from the fulcrum to the line joining the points of suspension of the scales. It is moreover necessary, that the centre of gravity of the balance and its load, should always lie below the fulcrum, in order that, if the instrument were displaced, it may return to the horizontal position; for if the centre of gravity lay above the fulcrum, it would have a tendency to upset, and if it coincided with the fulcrum, it would rest indifferently in any position.

In order to render this balance as nearly perfect as possible, it is necessary that it should indicate a small difference between the weight and its counterpoise, by a perceptible change from the horizontal position.



Suppose the line AB joining the points of suspension of the scales to pass through the fulcrum F, it should therefore be bisected at that point, and the centre of gravity of the equal weights will be at the fulcrum; let the weight of the instrument itself be W, and let its centre of gravity G be at a distance a below the fulcrum, and let the length of one of the equal arms AF be l, and let each of the equal weights in the two scales be P, then if a small additional weight w be placed in one scale, it should be indicated by a certain deflexion θ of the beam from the horizontal position; since the centre of gravity of the equal weights is at the fulcrum, they will be equilibrated in any position of the balance, and in order for equilibrium, it is necessary that the remaining forces W and w, acting at G and A, should balance one another; therefore, if perpendiculars

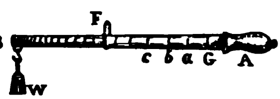
FH and FC be let fall from the fulcrum on the verticals passing through these points, w is to W as FH to FC, but FH is $a \sin \theta$, and FC is $l \cos \theta$, therefore w is to W in a ratio compounded of the ratios of a to l, and of the sine of θ to its cosine; whence it follows, that by diminishing W, the weight of the balance, or by diminishing the ratio of a to l, the deviation θ corresponding to a given difference of weight, or (as it is termed) the sensibility of the balance, is increased. In this construction of the balance, it is easy at once to ascertain the difference of the weights in the two scales, if the deflection θ be known, w the difference being proportional to the tangent of this angle; if the line AB did not pass through F, the deviation would cease to be an accurate measure, and the sensibility of the balance would then depend on the amount of the equal weights P in the two scales.



In the balance called the steelyard, the second of the above methods of producing equilibrium is employed; in this instrument the counterpoise P is a constant weight capable of being moved along the arm FA of the lever, which is divided into equal parts, as represented in the annexed figure. Let G be the centre of gravity of the steelyard, without either load or counterpoise, and w its weight, and let C be the point where the counterpoise P should be placed to balance the instrument when unloaded, then the product of P and FC is equal to the product of w and FG; and if 7 be the point where the counterpoise should be placed to balance a weight W suspended at B, the product of P and F7 is equal to the sum of the products of w and FG, and of W and FB, therefore, subtracting the former equals from the latter, there remains P multiplied by C7 equal to W multiplied by FB, therefore W is to P as C7 to FB. Let C1 be to FB as 1lb. to P, then a weight of one pound

in the scale will be counterbalanced by P placed at 1, and if c7 be 7 times c1, then c7 is to FB as 7 pounds to P, therefore W will be 7 pounds; or the number of pounds in the weight at B is equal to the number of divisions, each equal to c1, in the distance c7 of the counterpoise from the point C.

In the Danish balance equilibrium is produced by changing the place of the fulcrum; this balance is remarkably portable and simple in its construction; it is merely a straight rod with a weight A at one end, and a hook B at the other, and is suspended by a cord F, on which as a fulcrum it is to balance itself.



Let G be the centre of gravity of the instrument, it will therefore be in equilibrio with the fulcrum at this point when unloaded; let a weight W be suspended from the hook B, and cut BG, so that GF is to FB as W to the weight of the instrument, then F is the point where the fulcrum should be placed to make equilibrium with this weight on the hook, for the weight of the instrument multiplied by GF is its moment about the fulcrum F, and W multiplied by BF is the moment of the weight of W about the same point, and these being equal and opposite, will make equilibrium; therefore if w be the weight of the instrument, and if the line GB be cut at a, b, c, &c., so that Ga is to aB as 1lb. to w, Gb to bB as 2lb. to w, Gc to cB as 3lb. to w, &c., the points a, b, c will be the points by which the balance should be suspended to make equilibrium with weights of one, two, or three pounds on the hook B; it is evident that the intervals between these divisions will diminish rapidly as they approach the point B.

In the three kinds of balance we have hitherto considered, the weight of a body is ascertained by the adjustments necessary to produce equilibrium in a certain position; but in the bent lever balance, no adjustments of either the fulcrum or the counterpoise are required, but the weight is indicated by the position which the instrument assumes. The annexed

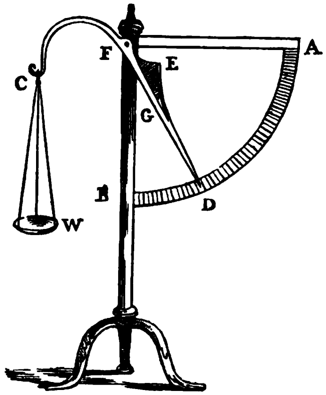


figure represents this kind of balance, AFB is a fixed stand bearing, on a fulcrum F, the bent lever DEC, to which is attached at C a scale, in which the body to be weighed is placed; there is also a quadrant ADB attached to the fixed stand, on which the index FD, passing through the centre of gravity G of the lever, moves; then, if there be no weight in the scale, the centre of gravity G will lie in the vertical passing through the fulcrum, and the index will point to B; if a weight W be placed in the scale, the position will be changed to that represented in the figure; let w be the weight of the lever, this is equivalent to a single force acting at its centre of gravity G, and the moment of this force about the fulcrum F must be equal to the moment of the weight W about the same point, that is, the product of w and FG multiplied by sine DFB, is equal to the product of W and FC multiplied by sine BFC, therefore W is to w in a ratio compounded of the ratios of FG to FC, and of the sine of DFB to the sine of BFC; if the lever be so constructed, that DFC is a right angle, then sine of BFC is cosine of DFB, and therefore W is to w in a ratio compounded of the ratios of FG to FC, and of the tangent of DFB to radius, therefore the weight W in the scale, is proportional to the tangent of this angle, or of the arc DB, which is its measure; and hence the weight may be immediately determined by the position of the index upon this arc. If the quadrant ADB be so graduated as to mark the positions of the index corresponding to weights of 1, 2, 3, &c., pounds, the tangents of the distances of these points from B will be proportional to the numbers 1, 2, 3, &c.; and as the arc does not increase in the same proportion with its tangent, these divisions will become smaller as they proceed from B toward A,

and no finite weight in the scale at C can ever bring the index to the point A, for AB being a quadrant, its tangent is infinite. If DFC be not a right angle, the proportions of the divisions will be different.

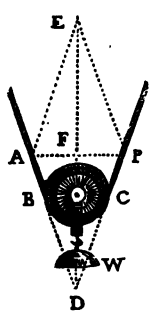
THE WHEEL AND AXLE.

The second of the mechanical powers is composed of two circular wheels of unequal diameters, and attached to an axis passing through their centres and perpendicular to their planes; the larger of these wheels is called the wheel, and the smaller the axle, the extremities of the axis pass through two circular holes in which they are capable of revolving; the power is usually applied in the direction of a tangent to the wheel and the weight in the direction of a tangent to the axle; these forces act on the wheel and axle either by means of flexible cords or bands, which pass round their circumferences, or by pressure on projecting pins or teeth, placed around the circumferences for that purpose. The conditions of equilibrium are the same as in the lever, for since the extremities of the axis rest on fixed points, the line joining these points is fixed, and since the power and weight act in parallel planes, there will be equilibrium when their moments about this fixed line are equal and opposite, that is, when the power, multiplied by the radius of the wheel, is equal to the weight multiplied by the radius of the axle; therefore, if R be the radius of the wheel, and r that of the axle, P is to W as r to R, and the weight is always greater than the power which counterbalances it; if a greater gain of power be required, it is easily obtained by a combination of several wheels and axles, so that the teeth of the first axle act on the teeth of the second wheel, and the teeth of the second axle on those of the third wheel, and so on, the power being applied to the first wheel, and the weight to the last axle; then if P', P'' represent the pressure of each axle on the next wheel, and $R \,.\, r$, $R' \,.\, r'$, $R'' \,.\, r''$,

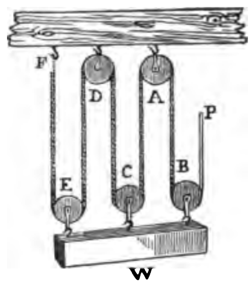
the radii of the successive wheels and axles, P is to P′ as r to R, P′ is to P″ as r' to R′, and P″ to W as r'' to R″, therefore, compounding these ratios, P is to W as $r\,r'\,r''$ to R R′ R″.

THE PULLEY.

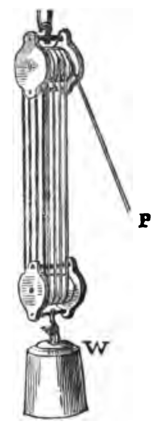
The pulley is a wheel capable of revolving about an axis fixed in a block, which is either fixed or moveable; it is used as a mechanical power by passing a flexible cord round the wheel, to one extremity of which the power is applied, the other extremity is usually either attached to the weight, or to a fixed point, or passes round another pulley. In the first case the weight is equal to the power, and the only mechanical advantage is a change of direction, for since both the power and weight act in the directions of tangents to the same circle, their moments about the centre of that circle (that is, about the axis of the pulley) will be equal when the forces are themselves equal. In the second case, if there be but one pulley, the weight is attached to its block, and will make equilibrium when it is equal and opposite to the resultant of the power and the tension of the cord AB; but since the force in the direction AB must be equal to that in the direction CP, their resultant bisects the angle contained by these lines, and is equal to DE, the diagonal of a parallelogram of which these forces are the sides; join AP, it will bisect DE in F, and be perpendicular to it, therefore P is to W as PD to 2DF, that is, as radius to 2 cos PDF; if PDA be 120° then PDF is 60°, and 2 cos PDF equal to radius, therefore in this case P is equal to W; if PDA be less than this, 2 cos PDF will be greater than radius, and W greater than P; and if PDA be cypher, that is, if the two



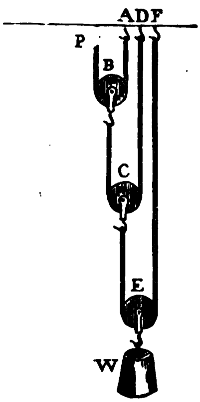
parts of the cord be parallel, W is equal to 2P, and this is the greatest ratio of W to P, or the greatest mechanical advantage to be gained by this application of the pulley. If the cord, instead of being attached to a fixed point at A, passed over a fixed pulley at A, and then passed under a second pulley C, also attached to the weight, and so on, passing alternately over any number of fixed pulleys, and under moveable ones attached to the weight W, and if the extremity be attached to a fixed point F, then it may be shewn as before, that all parts of this cord are equally strained, and that the weight W is equal to the resultant of twice as many forces equal to P as there are pulleys attached to the weight; therefore, if all the parts of the cord be parallel, and the number of these pulleys n, W will be $2n$ times P.



It is more convenient to place all the pulleys B, C, E, &c., on one axis, and A, D, &c., on another, as in the annexed figure, which facilitates the working of the pulleys considerably, but prevents the possibility of the cords being exactly parallel, therefore the actual gain of force is in this case something less than that given by the above equation; such combinations are called sheaves of pulleys.

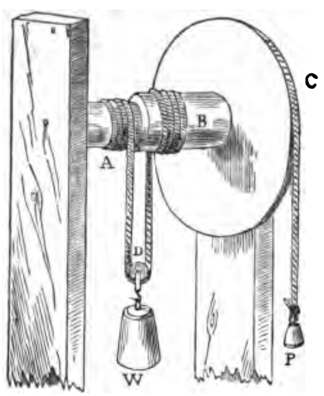


If instead of the block of the first moveable pulley B being attached to the weight, it be attached to a second cord which passes under the pulley C and is attached to a fixed point at D, and the block of the pulley C attached to a third cord which passes under the pulley E and is fixed at F, and so on, and the weight W is attached to the block of the last



pulley, then it is evident that W is equal to twice the tension of the cord EF, and that this is twice the tension of CD, and that tension again is twice P, W is therefore equal to 8P, and if there were any number n of pulleys in this combination, W is equal to P multiplied by the n^{th} power of 2. It is evident, that in place of single pulleys at B, C, E, sheaves of pulleys may be used, which will still further increase the mechanical advantage.

It would be easy to add to this list of the combinations of pulleys used in machinery, but the above are sufficient to explain the method of finding the conditions of equilibrium, and they are among the combinations most commonly used.

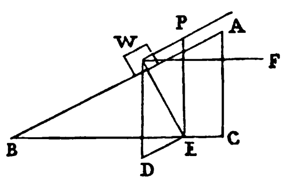


A combination of the pulley with the wheel and axle produces a very simple machine of great power. The axle consists of two parts, A and B, of different diameters, $2r$ and $2r'$, round one of which A a cord is coiled and brought under a pulley D, and then coiled in the same direction about the other part of the axle B, the weight W is attached to the block of this pulley, and the power P applied to the circumference of the wheel C in the same direction in which the weight acts on the smaller section of the axle at A, there will thus be three forces acting on the wheel and axle, namely, the power P, and the tensions of the cord at A and B, each of

which is $\frac{1}{2}$W; therefore there will be equilibrium when the sum of the moments P multiplied by R, and $\frac{1}{2}$W multiplied by r', is equal to the moment $\frac{1}{2}$W multiplied by r, therefore P multiplied by R is equal to $\frac{1}{2}$W multiplied by the difference between r and r', and P is to W as half the difference between r and r' to R, therefore the gain of power in this combination may be increased at pleasure by diminishing the difference between the radii of the parts A and B of the axle. This is the construction of the Chinese capstan.

THE INCLINED PLANE.

The next in order of the mechanical powers are those in which equilibrium is produced by means of the resistance of a perfectly smooth and hard surface, that is to say, a surface capable of equilibrating any force perpendicular to itself, but which offers no resistance to motion along the surface. The simplest mechanical power of this class is the inclined plane, that is, a perfectly smooth and impenetrable plane surface, inclined at any angle to the horizon, and used for the purpose of raising heavy bodies.



Let the plane of the figure ABC be perpendicular to the intersection of the inclined plane with the plane of the horizon, and let AB and BC be the right lines in which it intersects these two planes, and AC a vertical line, then AC is the height of the plane, BC its base, and AB its length. Now suppose a heavy body to be supported at W by a force WP, in a direction parallel to the plane, the weight of the body is equivalent to a single force WD in the direction of the vertical passing through its centre of gravity, and the resultant of these two forces is represented by WE, the diagonal of the parallelogram PWDE; therefore, if WE be perpendicular to the inclined plane, there is equilibrium; but in the triangles PWE, ACB, the angles P and A are

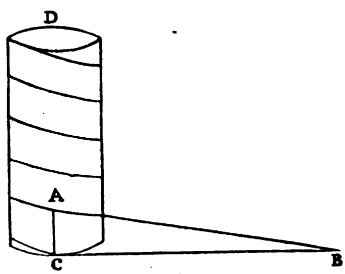
equal, and if W be a right angle it is equal to C, therefore the triangles are similar, and WP is to PE as AC to AB, that is to say, the power is to the weight as the height of the plane to its length.

If the body was supported by a power in the direction WF, parallel to the base of the plane, then the power, the weight, and the resistance of the plane, act in directions perpendicular to the three sides of the triangle ACB, therefore parallel to the sides of a triangle similar to it; therefore the power is to the weight as AC the height of the plane to BC its base. Chap. I., Cor. Prop. 3.

And in general, if the power P acted in any direction in the plane ABC, making an angle θ with AB, it may be resolved into two forces, P $\sin\theta$ and P $\cos\theta$, one perpendicular, the other parallel, to the plane; the former P $\sin\theta$ is equilibrated by the resistance of the plane, and the latter P $\cos\theta$ will be in equilibrio with the weight if P $\cos\theta$ is to W as AC to AB, therefore P is to W as AC to AB $\cos\theta$. Hence it appears, that the ratio of the power to the weight is the least possible when the power is parallel to the plane.

Since the weight and the resistance of the inclined plane both act in the plane ABC, their resultant is in the same plane, therefore, if the power did not act in this plane there could not be equilibrium.

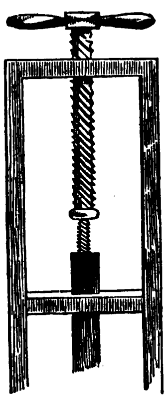
THE SCREW.



The screw may be considered as an inclined plane, AB, wrapped round a circular cylinder, the breadth of the plane being very small compared with the radius of the cylinder, and the base of the plane coinciding with the base of the cylinder; thus it is evident, that a spiral thread

will be described on the surface of the cylinder, whose inclination to the base of the cylinder will be at all points equal to that of the inclined plane; also the circumference of the cylinder being equal to the base of the inclined plane, the distance between two spires of the thread, measured in a direction parallel to the axis, will be equal to the height of the plane; and it is evident, that if the cylinder be placed with its base horizontal and its axis vertical, the conditions of equilibrium of a weight resting on the thread at any point, are the same as if it rested on this inclined plane; therefore, if the power act in a direction parallel to the base of the cylinder, P is to W as the distance between the threads to the circumference of the cylinder; but the weight does not usually rest on one point of the screw, but is distributed over a considerable portion of the thread by means of a second screw equal and similar to the first, and so placed, that they may be in contact throughout their length; for this purpose it is evidently necessary, that one spiral should be traced on a hollow cylinder, and the other on a solid cylinder of equal diameter; the former is usually called a nut, and the latter a screw; now if the nut were fixed with its axis vertical and a weight placed upon the screw, it is evident that this weight would be distributed over all the points at which the two threads touched one another, and if at each of these points a force were applied parallel to the base of the screw, there would be equilibrium if each of these forces were to the portion of the weight at the same point as the distance between two spires to the circumference of the cylinder, and therefore the sum of all the forces is to the whole weight in the same ratio, and the weight multiplied by the distance between two spires is equal to the sum of all these forces multiplied by the circumference of the cylinder; but it is evident from the construction of the instrument, that the axis of the screw must coincide with the axis of the nut, and therefore that it is a fixed line; and the effect of all these forces acting in planes perpendicular to the

axis, is the same as the effect of a single force acting also in a horizontal plane at the extremity of a lever or winch-handle, provided that the moment of this force is equal to the sum of the moments of the other forces (Chap. I., Prop. 19), that is to say, the weight will be equilibrated by a power acting perpendicularly to this lever, if the power multiplied by the length of the lever is equal to the sum of all the other forces multiplied by the radius of the cylinder; or, since the circumferences of circles are proportional to their radii, the power multiplied by the circumference of the circle described by the lever, is equal to the sum of all these forces multiplied by the circumference of the cylinder, that is (as was proved above) equal to the weight multiplied by the distance between two spires of the thread, therefore the power is to the weight as the distance between two spires of the thread to the circumference of the circle described by the lever.

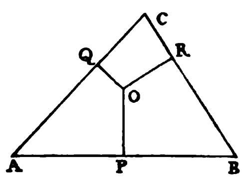


If two screws be turned, one on the inner, the other on the outer surface of a hollow cylinder, the former being provided with a solid screw, and the latter with a nut to fit them respectively, and the intervals between the spires being less in one screw than in the other, a machine may thus be formed of very great power,—it is called Hunter's screw. In this instrument the outer nut must be fixed, and the inner solid screw so constructed that it may be capable of moving in the direction of its length, but incapable of turning upon its axis, and the power is applied to turn the hollow cylinder by a lever, or winch-handle, the resistance or weight being opposed to the longitudinal motion of the solid screw; then there are three forces acting on the hollow cylinder tending to turn it on its axis, namely, the pressure of the weight on the thread of the inner screw, its pressure

on the thread of the outer screw, and the power; and it is necessary to equilibrium, that the sum of the moments of the two forces which tend to turn it in one direction, should be equal to the moment of that which tends to turn it in the opposite direction, that is to say, the power multiplied by the circumference of the circle described by the lever, together with the weight multiplied by the distance between the spires of the one screw, are equal to the weight multiplied by the distance between the spires of the other screw; therefore the power multiplied by the circumference of the circle described by the lever, is equal to the weight multiplied by the difference between the distances of the spires in the two screws, or the power is to the weight as the difference between these distances to the circumference of the circle described by the lever.

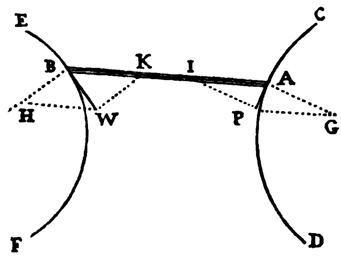
THE WEDGE.

The wedge is a triangular prism, that is to say, a solid bounded by three plane surfaces which intersect one another in three parallel lines, and by two equal triangles formed by two plane surfaces cutting the others perpendicularly; these triangles are the ends of the wedge, and the three other surfaces are denominated its back and its two faces. If the ends be isosceles triangles, the wedge is isosceles; and if their sides be unequal, it is a scalene wedge. The forces which are to make equilibrium by means of this instrument are, the power applied perpendicularly to its back, and the resistances or weights applied perpendicularly to its two faces. Now in order that these forces should make equilibrium, they must first meet in one point, and be in the same plane, and since the forces are perpendicular to the surfaces of the wedge, this plane must also be perpendicular to



them, and therefore parallel and similar to the two triangular ends. Let it cut the surface of the wedge in the sides of the triangle ABC; and let PO be the direction of the power, and QO, RO, the directions of the resistances meeting in the point O, then since the sides of the triangle ABC are perpendicular to these three forces, there will be equilibrium when they are proportional to the sides of this triangle (Chap. I., Cor. Prop. 3), therefore the power is to the sum of the weights as the back of the wedge AB to the sum of its faces AC and BC. If the wedge be isosceles, the pressures on the two equal faces must be equal to one another, and the power is to their sum as half the back of the wedge to one face; that is, as the sine of half the angle of the wedge to radius. If the wedge be scalene, the pressures on its faces must be to one another in the ratio of the lines AC and CB, therefore there cannot be equilibrium in this case unless these pressures be unequal.

Besides the mechanical powers which have been enumerated, there is another of frequent use in machinery, namely, a link or rod connecting two parts of a machine, which are, by the nature of the mechanism, capable of moving only in certain directions.



Let AB be the link, and CD, EF, the lines along which its extremities are capable of moving, and let AP, BW, be tangents to these lines at A and B; then it is evident, that any forces in the directions AG, BH, perpendicular to these tangents, will be equilibrated by the resistance of the fixed lines CD and EF. Let the power act in the direction AP, and the weight in the direction BW, and let them be proportional to these lines; through P draw PI and PG parallel to AG and AB, and through W draw WK and WH parallel to BH and AB; then the force AP is equivalent to the two forces AI and AG, and

the force BW is equivalent to the two forces BK and BH; but the forces AG and BH being perpendicular to the fixed lines, are equilibrated by their resistance, therefore there will be equilibrium if the remaining forces AI and BK are equal and opposite; but since API and BWK are right angles, AP is to AI as the cosine of BAP to radius, and BK (which is equal to AI) is to BW as radius to the cosine of ABW, therefore, *ex æquo*, the power AP is to the weight BW as the cosine of BAP to the cosine of ABW.

If the power and weight do not act in the directions of tangents to the lines CD and EF, they may be resolved into forces in the directions of AP and AG, BW and BH, of which AG and BH are equilibrated by the resistance of the fixed lines, and there will be equilibrium if the remaining forces AP and BW balance one another, that is, if AP is to BW as the cosines of the angles which they make with the direction of the link AB.

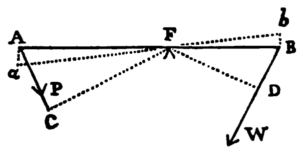
CHAPTER IV.

THE PRINCIPLE OF VIRTUAL VELOCITIES.

There is another principle from which the theory of equilibrium has been derived, which is neither so evident in itself, nor capable of such an easy demonstration as the principle of the composition and resolution of forces, but which has the advantage of being much more conveniently and directly applied to questions of the equilibrium of complicated machinery, and also to many of the most abstract questions of theoretical mechanics; this principle is called the principle of virtual velocities.

If there be a system of material points connected in any way so as to be capable of moving in certain directions, and if the points be supposed to move in any of these directions, they will in the first instant of the motion describe certain indefinitely small lines, which are called the virtual velocities of the points; for example, a lever is capable of turning on its axis, in which motion the two extremities will describe circles about this axis, and in the very first instant of their motion, they will describe indefinitely small arcs of these circles, which are to one another in the ratio of the radii; these small arcs are the virtual velocities of the extremities of the lever, and the virtual velocity of any other point in it is the indefinitely small space which that point moves through in the same time. Again, the virtual velocity of any of these points multiplied by the cosine of the angle which its direction makes with any given line, is said to be the "virtual velocity estimated in the direction of that line;" this will evidently vanish if the cosine of the angle be cypher, that is, if the angle be right, and it will be positive or negative according as the angle is acute or obtuse.

Now the principle of virtual velocities is, *that if any forces act on a system of material points (whether free or connected with one another or with other bodies in any manner whatever), and if the sum of the products of each force multiplied by the virtual velocity of its point of application estimated in the direction of that force be always equal to cypher, in every motion of which the system is capable, there will be equilibrium;* the sum being understood in the algebraic sense, as the difference between the sums of the positive and negative quantities. For a demonstration of this principle, see Appendix, Prop. 6.



Thus in the lever AFB, the virtual velocities Aa, Bb, of A and B, are proportional to the arms AF and BF, but they are also perpendicular to these lines, therefore the angles which they make with the directions of the forces AP and BW are equal or supplemental to the angles AFC and BFD contained by the arms of the lever and lines perpendicular to the forces, consequently the virtual velocity of A, estimated in the direction of AP, is to the virtual velocity of B estimated in the direction of BW, as AF multiplied by the cosine of AFC, to BF multiplied by the cosine of BFD, that is, as FC to FD, and the condition of equilibrium is, that P multiplied by FC be equal to W multiplied by FD, and that they be also of opposite signs, that is to say, that one of the angles PAa, WBb, be acute and the other obtuse; it is evident that this agrees with the condition of equilibrium proved in the preceding chapter.

Again, in the screw it is evident that by one revolution of the extremity of the lever to which the power is applied, the weight is raised through a space equal to the distance between two spires of the thread, and the ratio of any smaller motions, is the same as the ratio of the circumference of the circle described by the lever to the distance between these spires, there-

fore since the power and weight act each in the direction in which its point of application is capable of moving, the condition of equilibrium is, that the power multiplied by the circumference of the circle described by the lever, be equal to the weight multiplied by the distance between two spires of the screw, the same condition as has been already proved; and so in each of the other mechanical powers, the same conditions of equilibrium may be deduced from the principle of virtual velocities which have been already proved from the principle of the composition and resolution of forces.

In a complicated machine, where the power acts at one point of the combination and the weight at another, and where no other forces are applied, and these points are only capable of moving in the directions in which the forces are applied, this principle furnishes the following very simple formula for determining the condition of equilibrium, viz., the power multiplied by the virtual velocity of its point of application is equal and opposite to the weight multiplied by the virtual velocity of its point of application, or P is to W inversely as the velocities which their points of application would have if the machine were in motion; so that for every gain of power by machinery in motion, there is always a proportionate loss of velocity, even on the supposition that no other resistances were introduced by the very nature of the machinery itself, such as friction, &c., and consequently the highest perfection of which any machine for communicating motion is capable, is, that it should be as free as possible from such resistances; but this is a subject which belongs rather to the theory of motion than to that of equilibrium.

PART II.

DYNAMICS.

CHAPTER I.

DEFINITIONS AND PRINCIPLES.

The second branch of mechanics treats of the results which ensue when one or more forces acting on a body do not fulfil the conditions of equilibrium, and previously to entering on this investigation, it is necessary to lay down a few definitions and principles concerning motion; and first it is to be observed, that motion is either absolute or relative. Relative motion is a change of the position of one body with regard to others to which it is supposed to bear some relation. Absolute motion is the change of its actual position in space; thus if a number of bodies move in the same direction without changing their distances from one another, they will continue *relatively* at rest, though *absolutely* in motion; and if one of these bodies remain in the same place, while all the others move on, that one body will have changed its position with regard to the others, and so will be *relatively* in motion though *absolutely* at rest.

Now it is by its distances from certain other bodies, that we always determine the place of any particular object, and from a change in these distances, nothing but relative motion can be inferred; therefore it may at first sight appear, that we are incapable of determining whether the absolute motion belongs to the object itself, or to the other bodies to which it was referred; in other words, that it is impossible to dis-

tinguish between absolute and relative motion; we shall however find hereafter tests by which in some cases they may be distinguished, resulting from a consideration either of the cause or of the effect of the motion. The following propositions all refer to absolute motion.

Def. 1. The motion of a point is said to be uniform if it passes through equal spaces in equal portions of time. Therefore, when a point moves uniformly, the spaces passed through in different portions of time are proportional to these times.

Def. 2. The velocity of a point moving uniformly, is measured by the space which it passes through in a given time.

Cor. 1. In uniform motion the space passed through in any time is proportional to the product of the time and velocity; thus if two points A and B move with the uniform velocities V and v during the times T and t, the space passed through by A will be to that passed through by B as VT to vt, for the space passed through by A in the time T, is to that passed through by B in the same time as V is to v (Def. 2), and the space passed through by B in this time, is to that passed through in the time t as T is to t (Def. 1), therefore, compounding these ratios, the space passed through by A in the time T, is to the space passed through by B in the time t as VT to vt.

Cor. 2. Hence it is evident, that the velocity of a point is proportional to the space which it describes in any time divided by that time, for if S and s be the spaces described by the points A and B, then S is to s as VT to vt (Cor. 1), therefore V is to v as $\frac{S}{T}$ to $\frac{s}{t}$.

Hence the velocities of two points moving uniformly may be compared, although the times during which their motions are observed be different, for the ratio of the velocities is had by dividing each space by the time in which it is described.

Note. In algebraic computations, all quantities must be supposed to be represented by numbers, for which purpose it

is only necessary to assume some determinate quantity as unit, and the number of times that this quantity is contained in any other, is the number which represents that other quantity. Now if any quantities be arbitrarily assumed as the units of space and of time, and if S and T represent any other quantities of space and time, then the velocity of a point which describes the unit of space in the unit of time, is to the velocity of a point which describes the space S in the time T as $\frac{1}{1}$ to $\frac{S}{T}$, therefore if the former velocity be assumed as the unit of velocity, and if the latter be represented by V, this proposition may be thus expressed, $1 : V :: 1 : \frac{S}{T}$, therefore $V = \frac{S}{T}$; thus, if one foot be assumed as the unit of space, and one second as the unit of time, and the velocity of one foot per second as the unit of velocity; then the velocity of one hundred yards per minute will be represented by the quotient $\frac{300}{60} = 5$; for one hundred yards, being equal to three hundred feet, is represented by the number 300, and one minute, being sixty seconds, is represented by 60; it is to be observed, that the units of time and space are perfectly arbitrary, except that, when they are once agreed on, they must of course continue invariable throughout the calculation; but in order that the equation $V = \frac{S}{T}$ may be true, it is necessary that the velocity assumed as unity should be the velocity of a point which passes through the unit of space in the unit of time.

Def. 3. The velocity of a point which does not move uniformly, is measured at each instant by the indefinitely small space described in that instant, divided by the indefinitely short time in which it is described; or, more accurately, it is the limit to which this quotient is found to approach indefinitely when the time is diminished indefinitely.

Axiom 1. A particle of matter, if at rest, will remain for

ever at rest, or if in motion, will continue for ever to move uniformly in a right line, unless it be acted on by some force.

Axiom 2. If a particle of matter at rest be acted on by any force, the velocity produced will be proportional to, and in the direction of that force.

N. B.—If any number of forces act at once on a rigid body, they produce the same velocity as would have been produced by their resultant.

Def. 4. Two particles are said to contain equal quantities of matter, if equal forces produce equal velocities in each. The resistance that a body offers to any attempt to put it in motion, is called the "*vis inertiæ*" of the body, hence the definition is equivalent to this, that the quantities of matter in two particles are equal if they have equal "*vis inertiæ.*"

Def. 5. The quantity of motion in a body is the product of its velocity multiplied by its quantity of matter. Hence if two bodies move with equal velocities, their quantities of motion are proportional to the quantities of matter which they contain.

Def. 6. Forces which produce equal velocities in different bodies, are called equal accelerating forces.

Def. 7. The force which is capable of causing a particle at rest to move with any velocity, is called the force due to that velocity.

Axiom 3. If a particle of matter in motion be acted on by any force, the resulting velocity will be the same as that which would have been produced in the same particle, if at rest, by the resultant of this force and the force due to the velocity with which it had been moving.

Hence, if the force be in the direction of the motion, the resulting velocity will be equal to the sum of the original velocity and that which would have been produced by the force if the particle had been at rest. And if the force act in the opposite direction, the resulting velocity will be equal to the difference of these two velocities.

N. B.—This axiom and the second are usually comprised in one which is called the second law of motion, and is thus expressed: "Motion, or change of motion, is in the direction of the force impressed and proportional to it." This law is here divided into two, for the purpose of avoiding any difficulty in understanding the phrase "change of motion."

Axiom 4. The weights of equal particles of matter are equal.

This axiom is easily verified, by allowing different bodies to fall freely in vacuo, they will all be found to move with equal velocities; therefore, if they were divided into any number of equal particles, all these particles must have been acted on by equal forces (Def. 4); but the force which acts on each particle is its weight, therefore the weights of equal particles are equal.

Axiom 5. Action and reaction are equal and opposite(*a*).

Prop. 1. A force which passes through the centre of gravity of a rigid body will cause all its parts to move with equal velocities in parallel directions.

Let the body be divided into particles containing equal quantities of matter, and let equal parallel forces be applied

(*a*) The fifth axiom, which was called by Newton the third law of motion, has been more generally stated by D'Alembert thus: If any forces act on different points of a body, or system of bodies, the motion of these points may, in consequence of their mutual connexion, be different from those which would have taken place if they were unconnected; but if forces be applied to all the particles equal and opposite to the forces due to the velocities actually produced, these forces will make equilibrium with the forces which produced them. In Newton's statement of the third law of motion, the change of motion in any particle caused by its connexion with another, is called the action of that particle; and the corresponding change in the second particle is called the reaction; and since these two are equal and opposite, the forces to which they are due will make equilibrium; and since this is true of all the particles, the forces due to all the changes of motion throughout the system make equilibrium; but these forces are all evidently equal and opposite to the resultants of the forces between which there is equilibrium, by the principle of D'Alembert; whence it is evident that the two principles are identical.

to each of these particles, then the body will move in the manner required (Def. 4); but the resultant of these forces is a single force parallel to them, equal to their sum, and passing through the centre of gravity of the body, and this force will produce the same effect.

Schol. Hence a force whose direction passes through the centre of gravity of a body, will not produce rotatory motion, unless the body be restrained by the resistance of a fixed point, or by some other force not passing through the centre of gravity.

Prop. 2. If the forces P and Q acting on the centres of gravity of the bodies A and B cause them to move with equal velocities, then P is to Q as A to B.

For let the body A be divided into any number m of equal particles, and let B contain n particles equal to them, then A is to B as m to n, but the force P is equal to the sum of m equal forces acting on its particles, and Q is equal to the sum of n forces equal to them (Def. 4), therefore P is to Q as m to n, that is, as A to B.

Schol. The forces P and Q are equal accelerating forces (Def. 6), therefore it follows, that equal accelerating forces are proportional to the quantities of matter on which they act. Now it has been already stated, that gravity produces equal velocities in all bodies, therefore the weights of different bodies are proportional to the quantities of matter they contain.

Prop. 3. If forces F and G, acting on the centres of gravity of the bodies A and B, produce the velocities P and Q, then F is to G as AP to BQ.

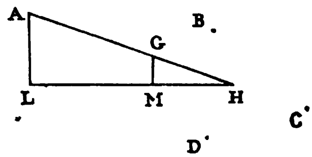
For let a third force X acting on B produce the velocity P, then F is to X as A to B (Prop. 2); but X is to G as P to Q (Ax. 2), therefore F is to G as AP to BQ.

Hence, if the forces are equal, AP is equal to BQ, that is to say, equal forces produce equal quantities of motion (Def. 5).

Prop. 4. A force acting on any part of a system of par-

ticles of matter, whether unconnected or connected in any manner, will cause the centre of gravity of the system to move as if all the particles were collected at that point, and an equal force applied to it in a parallel direction.

First, let the particles be unconnected, and let a force F act on the particle A, in the direction AL; let H be the centre of gravity of the remaining particles B, C, D, &c., of the system, join AH, and cut it at G, so that AG is to GH as the sum of the remaining particles to A, then G is the centre of gravity of the system; but since the particles are supposed to be unconnected, the force acting on A will make no change in the position of the other particles, therefore their centre of gravity H will remain unmoved while the particle A moves in the direction AL; let it move in a given time to L, join HL, and cut it at M, so that LM is to MH as AG to GH, then M is the new position of the centre of gravity, and it has therefore moved from G to M, while the point A moved to L, therefore GM is to AL as the velocity of the centre of gravity to the velocity of A; but since AG is to GH as LM to MH, GM is parallel to AL, and GM is to AL as GH to AH, that is, as A to the sum of all the particles; therefore the velocity of the centre of gravity is to the velocity of A, as A is to the sum of all the particles; and A multiplied by the velocity of A, is equal to the sum of all the particles multiplied by the velocity of the centre of gravity, therefore the force which produced this velocity in A is equal to the force which would produce the velocity of the centre of gravity if all the particles were collected there (Prop. 3), and it is also parallel to it, since AL is parallel to GM.



Secondly. Let the particles be connected in any manner, the only effect of such connexion is, that they will exert a certain action on one another, which is always accompanied

by an equal and opposite reaction (Ax. 6), and the motions of all the particles will be the same as if they were unconnected, and forces equal to these actions and reactions applied to them; but each of these forces will produce the same motion in the centre of gravity as if it were directly applied to that point (Case 1), and therefore all the forces will produce the same effect on the motion of the centre of gravity as if their resultant were applied to it; but since action and reaction are equal and opposite, they will have no resultant, and therefore will cause no change whatever in the motion of the centre of gravity, which will therefore move as if the particles were unconnected, that is (Case 1), as if all the particles were collected at that point, and a force equal and parallel to the given one applied to it.

Cor. 1. It is evident that if the force were applied in a parallel direction to any other particle of the system, it would produce the same motion in the centre of gravity.

Cor. 2. If several forces were applied to different particles in the system, the motion of the centre of gravity would be the same as if equal and parallel forces were applied to all the particles collected at the centre of gravity.

Cor. 3. If two equal parallel forces act in opposite directions, on different particles of a system, the centre of gravity will remain at rest; and, therefore, if the system be a rigid body, the only motion which can result is rotation about this point.

Cor. 4. If a force act on a rigid body in a direction which does not pass through the centre of gravity, it will produce a double motion, rotatory and progressive; the former the same as would take place if the centre of gravity were fixed; the latter, the same as if the force had acted directly on the centre of gravity; that is to say, the centre of gravity will move in a right line parallel to the direction of the force, with the same velocity as if the force had acted in that line, and the other

points of the body will rotate about the centre of gravity, performing their revolutions in the same time as if the centre of gravity had been fixed. The first part has been already proved, and the second is evident from the preceding corollary: for if an equal parallel force acted on the centre of gravity in an opposite direction it would have no effect on the rotatory motion (Sch. Prop. 1), but it would cause the centre of gravity to remain at rest (Cor. 3), therefore the rotatory motion would be the same as if this point were fixed.

From this last proposition and its corollaries, it appears, that in order to ascertain the motion of the centre of gravity of a body, it is only necessary to ascertain the motion of a single particle or material point acted on by forces equal and parallel to those which act on the given body, for if this point contain the same quantity of matter as the given body, its motion will be precisely the same as that of the centre of gravity; thus, if the force be a single impulse acting on a body at rest, its centre of gravity will move uniformly in a right line parallel to the direction of the impulse, and with a velocity proportional to the force directly, and to the quantity of matter inversely. If there be several simultaneous impulses, the motion of the centre of gravity will be the same as if the resultant of a number of forces equal and parallel to these acted on this point. And if the body, instead of being at rest had been in motion at the time of the impulse, the motion of the centre of gravity afterwards will be in the direction of and proportional to the resultant of a force equal and parallel to the impulse, and of the force due to the previous velocity of the centre of gravity. If the force, instead of being a single impulse, were a continued force, it may be considered as a multitude of impulses acting in immediate succession on a body in motion, and, therefore, produces the same motion of its centre of gravity as equal and parallel forces acting directly on a material point of equal weight would produce, therefore what-

ever is proved in the following propositions of the motion of a material point, is equally true of the motion of the centre of gravity of any body, whatever be its form or magnitude, if it be acted on by the same, or by equal and parallel forces; always excepting the case of a body in any way restrained by external obstacles, for any such restraints are themselves forces whose intensity and direction generally depend on the form of the body; an instance of this exception will appear hereafter in the motions of pendulums.

CHAPTER II.

OF RECTILINEAR MOTION.

Definition.—A constant or uniform force is a force which acting constantly produces equal velocities in equal portions of time.

Prop. 1. If any number of forces act in the same direction, either simultaneously or successively on a particle originally at rest and free, its velocity will be equal to the sum of the velocities which would have been produced by each of the forces acting on it separately.

For its velocity is equal to that which would have been produced by the resultant of all the forces (Ax. 3), therefore it is to the velocity which would have been produced by any of the forces separately, as the resultant is to that force (Ax. 2); and therefore it is to the sum of all the velocities which would have been produced by each force acting separately, as the resultant is to the sum of all the forces; but since all the forces act in the same direction, their resultant is equal to their sum, therefore the velocity produced by it is equal to the sum of the velocities which would have been produced by each of the forces acting separately.

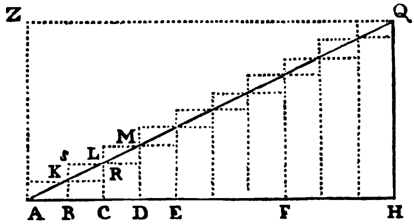
Prop. 2. If a particle, originally at rest, be acted on by an uniform force, its velocity at the end of any time will be proportional to that time.

For suppose the time to be divided into a number of equal portions of a given length, then the velocities which would have been produced in each of these portions are equal to one another, and their sum is proportional to their number, that is, proportional to the time; but the velocity produced in the entire time is equal to the sum of these velocities (Prop. 1); therefore it is proportional to the time.

Prop. 3. If different uniform forces F and f act on different particles a, b, originally at rest, their velocities after the times T, t, will be to one another as FT divided by a, to ft divided by b.

For the velocity of a after the time T, is to the velocity of b after the same time as F divided by a to f divided by b (Chap. I. Prop. 3); but the velocity of b after the time T is to its velocity after the time t as T to t (Prop. 2); therefore, compounding these ratios, the velocity of a after the time T, is to the velocity of b after the time t as FT divided by a to ft divided by b.

Prop. 4. To exhibit by geometric construction the spaces passed over in given times by a particle accelerated by an uniform force.



Let the particle be originally at rest, and let the time be divided into any number of equal parts, and let the same number of equal portions AB, BC, &c., be taken on the right line AH, then any portion AF of this line may be understood to represent a portion of time containing the same number of these equal intervals that AF contains of the equal parts AB, BC, &c., these lines being evidently proportional to the times which they thus represent. Let AH then represent the entire time of the motion, erect the perpendicular HQ, and join AQ, meeting the perpendiculars BK, CL, &c., erected at the points of division of the right line AH, then these perpendiculars being proportional to the lines AB, AC, &c., are proportional to the times represented by these lines, and are therefore proportional to the velocities of the particle at the ends of these times (Prop. 2), therefore these perpendiculars may be taken to represent the velocities; now if the velocity BK continued uniform during the interval of time BC, the space described in that time would be the product of the time and velocity

(Def. 2, Cor. 1), therefore it would be represented by the rectangle BKRC; and if the velocity CL continued uniform from B to C, the space described would in like manner be represented by BSLC; but the velocity between B and C is greater than BK and less than CL; therefore the space actually described in the interval BC is represented by a quantity greater than one rectangle and less than the other, and the same is true of the other intervals, therefore the entire space described in the time AH is greater than the sum of all the rectangles CK, DL, EM, &c., which lie within the triangle AHQ, and less than the sum of the rectangles AK, BL, CM, &c., which exceed the same triangle, and this is true however small the intervals AB, BC, &c., may be; but it is evident, that the only space which is always greater than the sum of the inner, and less than the sum of the outer rectangles, is the triangle AHQ itself, therefore this triangle represents the space described in the time AH.

Prop. 5. The same things being supposed as in the last proposition, the space described in any time beginning at the commencement of the motion, is one-half of that which would have been described in the same time if the final velocity had continued uniform throughout the motion.

For the time being represented as before by the right line AH, and the final velocity by HQ, the space described in that time is represented by the triangle AHQ (Prop. 4); but if the velocity HQ had continued uniform throughout the motion, the space described would have been represented by the product of this velocity and the time (Def. 2, Cor. 1), that is by the rectangle AZQH; but the triangle AHQ is one-half of this rectangle, therefore the space described is one-half of that which would have been described, if the final velocity had continued uniform throughout the motion.

Prop. 6. If two particles be moved from a state of rest, each by an uniform force, the spaces described by them in any times, reckoning from the beginning of their motions, are

to one another as the products of the times and final velocities.

For the spaces which would have been described in these times, if the final velocities had continued uniform, are in this ratio (Def. 2, Cor. 1), and the spaces actually described are the halves of these (Prop. 5), therefore they are in the same ratio.

Prop. 7. The same things being supposed, the spaces described are proportional to the squares of the times multiplied by the forces, and divided by the quantities of matter moved.

For the final velocities are as the forces multiplied by the times, and divided by the quantities of matter (Prop. 3); and the spaces are to one another as the products of these final velocities and the times, that is, as the products of the forces multiplied by the squares of the times, and divided by the quantities of matter moved.

Cor. Hence the spaces described by the same particle are proportional to the squares of the times since the commencement of the motion.

Scholium. If two equal weights be connected by a cord which passes over a pulley as nearly as possible free from friction and from other impediments to motion, and if a small additional weight be attached to one of them, a motion will ensue sufficiently slow to be capable of being easily observed with tolerable accuracy. Suppose then that the additional weight was so attached, that it could be removed at any time without checking the motion, it is evident that the weights being in equilibrio will continue to move uniformly (friction and other resistances being supposed to produce no perceptible effects), therefore the velocity is measured by the space passed through in a given time (Def. 2). This is in fact the construction of the instrument commonly known as Attwood's machine. Now it is found by the most accurate experiments,

that the space which is thus passed through in any given time (as one second) after the additional weight is removed, is exactly proportional to the time during which this weight had been attached, that is to say, the velocity at any instant is proportional to the time since the beginning of the motion, which we have already seen to be the case when a particle of matter is accelerated by an uniform force, whence it may be easily inferred, that the accelerating force in this experiment is an uniform force, and it is evident that the same instrument may be easily applied to verify the other propositions which have been proved concerning such forces; it may also be observed, that if the additional weight be increased in any ratio (the whole weight remaining the same), the velocity produced in a given time will be increased in the same ratio; therefore, if the whole weight be given, the accelerating force is proportional to the excess of the descending over the ascending weight; whence it appears, that the accelerating force is to the force of gravity as this excess is to the entire weight, from which it follows, that gravity is also an uniform force; and its intensity, or the velocity which it is capable of producing in a given time, may be computed by means of this instrument; but this observation is not susceptible of so great accuracy as that of the pendulum, which will be hereafter described.

Prop. 8. The same things being supposed as in Prop. 6. The squares of the final velocities are as the spaces passed through, multiplied by the forces and divided by the quantities of matter moved.

For the square of the velocity is equal to the product of the time and velocity multiplied by the velocity and divided by the time; but the product of the time and final velocity is proportional to the space (Prop. 6), and the velocity divided by the time is proportional to the force divided by the quantity of matter (Prop. 3), therefore the square of the final velocity is proportional to the space multiplied by the force and divided by the quantity of matter.

Cor. If two particles fall down different heights, the squares of their final velocities will be proportional to the heights.

For the forces are in this case proportional to the quantities of matter, therefore the ratio is unaltered by multiplication by one and division by the other.

Prop. 9. If a particle be moved from a state of rest by an uniform force, the spaces passed through in successive seconds of time will be proportional to the odd numbers 1, 3, 5, &c.

For the space passed through in the first second is to that passed through in the first two seconds as one to the square of two (Cor. Prop. 7), and it is to that passed through in the first three seconds as one to the square of three, and so on; therefore, if the space passed through in the first second be a, that passed through

in the first two seconds will be	$4a$,
in the first three seconds . .	$9a$,
in the first four seconds . . .	$16a$,
&c.	&c.;

and subtracting from each of these spaces the one immediately preceding, we have for the space passed through

in the first second . .	a,
in the second . . .	$3a$,
in the third	$5a$,
in the fourth . . .	$7a$,
&c.	&c.;

quantities which are proportional to the odd numbers 1, 3, 5, 7, &c.

It is evident, that the same proposition is true of any other equal portions of time reckoned from the beginning of the motion.

Prop. 10. If a particle of matter be projected in a vertical direction with any given velocity v, and continue to move during a given time t, the change of its velocity in that time

will be equal to the velocity which it would have acquired in the same time if it had fallen from a state of rest.

For let the velocity v be that which would have been produced by the force F, and let f be the resultant of the forces of gravity which act on the particle during the time t, then the resultant of all these forces will be the sum or difference of F and f, according as the particle was projected downwards or upwards, and the velocity of the body at the end of the time t is equal to that which would have been produced by this force (Ax. 4), and therefore it is to the original velocity as the sum or difference of F and f is to F, therefore the change of velocity is to the original velocity as f to F; but the velocity which it would have acquired in the same time, if it had fallen from a state of rest, is equal to that which would have been produced by the force f, therefore this velocity is also to the original velocity as f to F, and therefore it is equal to the change of velocity produced in the same time.

Cor. 1. If the particle be projected vertically upwards, and if F be equal to f, the velocity will be destroyed.

Cor. 2. If a particle fall from a state of rest until it acquire a certain velocity v, and if another particle be projected upwards with an equal velocity, it will come to a state of rest in an equal time. For in this case it is evident, that F is equal to f, each of these being the force of gravity acting for the same time.

Cor. 3. If a particle be projected vertically upwards with any velocity v, the height to which it will ascend is equal to that from which it should have descended to acquire an equal velocity.

For the time of ascent is equal to the time of descent (Cor. 2), and it is evident, that the velocity remaining in the ascending particle at any time before the end of the ascent, is equal to that acquired by the descending one at an equal time from the beginning of the descent; therefore the spaces moved through in corresponding equal portions of time are equal,

and therefore since the whole times are equal the whole spaces must be so also.

Note. This height is called the "height due to the velocity."

Prop. 11. If a particle be projected vertically downwards with any velocity, and continue to move through a given space BC, the difference between the squares of the velocities at B and at C will be equal to the square of the velocity which it would have acquired in falling through the same space from a state of rest.

For let AB be the height due to the velocity of the particle at the point B, then if a particle be supposed to fall from a state of rest at A, it will at B have the same velocity as the given particle, and will be equally accelerated in passing from B to C, therefore its velocity at C will also be equal to that of the given particle; but the square of this particle's velocity at B is to the square of its velocity at C, as AB to AC (Cor. Prop. 8); therefore the square of the velocity at B is to the difference of the squares of the velocities at B and at C as AB to BC; but the square of the velocity at B is to the square of the velocity which a particle would acquire in falling from B to C, as AB to BC; therefore the square of this velocity is equal to the difference of the squares of the velocities of the given particle at B and at C.

Cor. If two bodies be accelerated or retarded in their motion through a given space by the action of a given uniform force, that which had the greater velocity at the beginning of the motion will be less accelerated or retarded than the other.

For let V, U be their velocities at the beginning of the motion, and v, u their velocities at the end, then since the space is the same, the difference of the squares of V and v is equal to the difference of the squares of U and u, that is, the product of the sum and difference of V and v is equal to the product of the sum and difference of U and u; therefore if the

sum of V and v be greater than the sum of U and u, the difference between V and v is less than the difference between U and u; that is, the body which moved faster is less retarded or accelerated than the other.

Prop. 12. If a body fall down an inclined plane without any resistance from friction or any other cause, the accelerating force is to the force of gravity as the height of the plane to its length.

For the moving force is equal to that which, acting in the opposite direction up the plane, would make equilibrium; therefore it is to the weight of the body as the height of the plane to its length (Part I. Chap. III.); but the moving force is to the weight of the body as the accelerating force to the force of gravity; therefore these forces are also in the ratio of the height to the length.

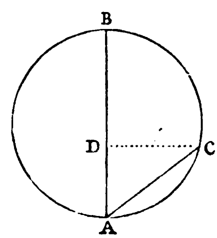
Prop. 13. If two bodies begin to move at the same time, one down an inclined plane, the other vertically, the velocity of the first is to the velocity of the second at any time as the height of the plane to its length.

For these velocities are in the ratio of the forces (Ax. 2), that is, as the height of the plane to its length (Prop. 12).

Prop. 14. The same things being supposed, the spaces passed through by the two bodies will be in the ratio of the height of the plane to its length.

For the space described by each body is one-half of that which would be described in the same time with its final velocity continued uniform; therefore these spaces are in the ratio of the final velocities, that is, as the height of the plane to its length.

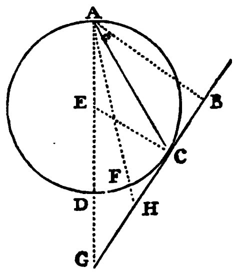
Cor. 1. If a circle ABC be described in a vertical plane, and any chord AC be drawn terminated at the lowest point A of the circumference, a body will roll down this chord in the same time that it would fall down the vertical diameter AB.



For draw the horizontal line CD,

then AC is an inclined plane whose height is AD and length AC, but AC is to AB as AD to AC; therefore the spaces AB, AC will be passed through in the same time (Prop. 14). The same is evidently true of chords terminated at the upper point B.

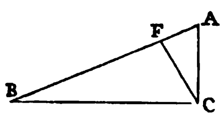
Cor. 2. Hence may be found the right line along which a body will descend in the shortest possible time from a given point to a given inclined plane.



Let A be the given point, AG a vertical line passing through this point, and AB a perpendicular to the inclined plane, the right line AC which bisects the angle BAG, will be the required line.

For draw CE parallel to AB, and cutting AG in E, then since AB is perpendicular to the inclined plane, CE is so also, and the angle ACE is equal to the alternate CAB, and therefore equal to CAE; therefore CE is equal to AE; with E as centre, and EA or EC as radius, let a sphere ACD be described, it will touch the plane at C, since EC is perpendicular to it, and the time of falling down the chord AC will be equal to the time of falling down the vertical diameter AD (Cor. 1); but this time is also equal to the time of falling down any other chord AF, and therefore less than the time of falling down AH, since the point H is necessarily outside the circumference of the circle of which AD is the diameter.

Prop. 15. If a particle fall down an inclined plane AB, its final velocity is equal to that which would have been acquired by falling down its vertical height.



Let fall CF perpendicular on AB, then since the particle is accelerated by an uniform force, the square of its velocity at F is to the square of its velocity at B as AF to AB; but AF is to AC as AC to AB, therefore AF is to AB as the square of AC to the square of AB, therefore the velocity at

F is to the velocity at B as AC to AB; but if another particle were let fall from A down AC at the same time that the given particle begins to fall down AB, it will reach C at the same time that the given particle reaches F (Cor. 1, Prop. 14), therefore the velocity at F will be to the velocity of the second particle at C, as AC to AB (Prop. 13), but the velocity at F is to the velocity at B in the same ratio, therefore the velocities at B and at C are equal.

Cor. Hence if particles fall down different inclined planes of the same vertical height, their final velocities are equal.

Prop. 16. The time of falling down an inclined plane is to the time of falling down its vertical height as the length to the height.

For the final velocities are equal (Prop. 15), and the times of falling are double the times of moving through the same spaces with these velocities continued uniform (Prop. 5).

Note to Chapter II.

It follows from the third Axiom, that if a particle in motion be acted on by any force, the quantity and direction of the resultant velocity may be derived from the quantities and directions of the component velocities, in the same way that the resultant of any number of forces is found; thus if a particle, moving with a velocity which is represented in quantity and direction by one side of a parallelogram, be acted on by a force capable of producing in an equal particle at rest a velocity represented in quantity and direction by the adjacent side, the resultant velocity will be represented in quantity and direction by the diagonal of the parallelogram; and in like manner, all the relations which have been proved to exist between component forces and their resultants exist also between the velocities produced by these forces.

Now suppose a particle, whose coordinates referred to rectangular axes are at any instant x, y, z, to be moving with a given velocity v, in a direction which makes angles α, β, γ with lines parallel to the three axes, then the components of this velocity parallel to the axes are $v\cos\alpha$, $v\cos\beta$, $v\cos\gamma$. Let this particle be acted on continually by any force, and let the intensity of this force during the indefinitely short interval of time dt be such, that if it continued to act uniformly in the same direction during a time equal to

unity, it would produce in the given particle a velocity F, then F is the measure of the accelerating force, and the velocity which it is capable of producing in the time dt is $\text{F}dt$; let this velocity be resolved into three parallel to the three axes, they will be $\text{F}\cos\alpha' dt$, $\text{F}\cos\beta' dt$, $\text{F}\cos\gamma' dt$, where α', β', γ' are the angles which the direction of the force F makes with lines parallel to the three axes of coordinates; and the velocities parallel to these three axes at the end of the time dt will be $v\cos\alpha + \text{F}\cos\alpha' dt$, $v\cos\beta + \text{F}\cos\beta' dt$, $v\cos\gamma + \text{F}\cos\gamma' dt$, or (if X, Y, Z be the components of the accelerating force F parallel to these three axes, and a, b, c the components of the velocity at the beginning of the time dt, and $a + da$, $b + db$, $c + dc$, the components of the velocity at the end of this time) $a + da = a + \text{X}dt$, $b + db = b + \text{Y}dt$, $c + dc = c + \text{Z}dt$, therefore

$$\frac{da}{dt} = \text{X}, \quad \frac{db}{dt} = \text{Y}, \quad \frac{dc}{dt} = \text{Z}, \tag{1}$$

in which equations the first members are evidently the differential coefficients, taken with regard to the time, of the velocities parallel to the three axes; and the second members are the components of the accelerating force parallel to the same axes. Moreover, since v is the velocity of the particle at this instant of time, vdt is the space through which it will move in the time dt, and $vdt\cos\alpha$, $vdt\cos\beta$, $vdt\cos\gamma$, will be the quantities by which its coordinates x, y, z are increased in this time, that is, $dx = vdt\cos\alpha = adt$, $dy = vdt\cos\beta = bdt$, $dz = vdt\cos\gamma = cdt$, or

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b, \quad \frac{dz}{dt} = c; \tag{2}$$

and, differentiating these equations,

$$\frac{d^2x}{dt^2} = \frac{da}{dt}, \quad \frac{d^2y}{dt^2} = \frac{db}{dt}, \quad \frac{d^2z}{dt^2} = \frac{dc}{dt};$$

eliminating the velocities a, b, c, between these equations and equations (1) there results

$$\frac{d^2x}{dt^2} = \text{X}, \quad \frac{d^2y}{dt^2} = \text{Y}, \quad \frac{d^2z}{dt^2} = \text{Z}; \tag{3}$$

from which equations the position of the particle at every instant may be determined by two integrations, if the accelerating forces X, Y, Z be known.

Now let $b = 0$, $c = 0$, $\text{Y} = 0$, $\text{Z} = 0$, therefore $a = v$, $\text{X} = \text{F}$, and the particle is moving in a direction parallel to the axis OX, and the accelerating force acts in the same direction; in this case equations (1), (2), and (3), become

$$\frac{dv}{dt} = \text{F}, \text{ (a)}, \qquad \frac{dx}{dt} = v, \text{ (b)}, \qquad \frac{d^2x}{dt^2} = \text{F}, \text{ (c)};$$

and multiplying equations (a) and (b) together,

$$vdv = \text{F}dx, \quad \text{(d)},$$

an equation which is generally more easily integrated than the others, in consequence of its not containing the element of time dt.

For example, let the force be constant, $F = \mu$, then equation (a) becomes $\frac{dv}{dt} = \mu$, therefore $v = \mu t + v'$, where v' is the initial velocity; substituting this value for v in equation (b); $\frac{dx}{dt} = \mu t + v'$, therefore $x = \frac{\mu}{2}t^2 + v't + x'$; also equation (d) becomes $vdv = \mu dx$, therefore $v^2 = 2\mu x + e$. Now suppose for more simplicity, that the position of the particle when $t = 0$ is the origin of coordinates, and the two latter integrals become $x = \frac{\mu t^2}{2} + v't$, $v^2 - v'^2 = 2\mu x$; these equations contain the whole theory of the motion of a body falling in vacuo by the force of gravity, when we substitute for μ the accelerating force of gravity, that is 32 feet 2 inches (1 second being taken as the unit of time); the same equations determine the motion down an inclined plane, if this value of μ be multiplied by the height and divided by the length of the plane; it is easy to see that they are identical with the propositions in Chapter II.

For a second example, let the force be directed to a given point in the line of the particle's motion, and proportional to the distance from that point, then the given point may be taken as the origin of coordinates, the direction of the particle's motion being as before the axis OX; then if x be the distance of the particle from the origin at any instant $F = -\mu x$, the sign is negative, because the force, being directed to the origin, tends to diminish the distance x. Hence equation (d) becomes $vdv = -\mu x dx$, and integrating $v^2 - v'^2 = \mu (x'^2 - x^2)$ where v is the velocity at the distance x and v' the velocity at the distance x', let $v'^2 + \mu x'^2 = \mu a^2$, therefore $v^2 = \mu (a^2 - x^2)$, whence it appears that x cannot exceed the limits $\pm a$, and that when $x = \pm a$, $v = 0$; again, substituting its value for v in equation (b), $\frac{dx}{dt} = \sqrt{(\mu (a^2 - x^2))}$, therefore $dt = \frac{1}{\sqrt{\mu}} \frac{dx}{\sqrt{(a^2 - x^2)}}$; or if $x = a \sin\theta$, $dt = \frac{1}{\sqrt{\mu}} \frac{a.d(\sin\theta)}{a\cos\theta} = \frac{1}{\sqrt{\mu}} d\theta$, and integrating $t = \frac{1}{\sqrt{\mu}}\theta + c$, therefore $\theta = \sqrt{\mu}(t - c)$ and $x = a.\sin\sqrt{\mu}(t - c)$; let the time commence when the particle is at the origin, then $0 = A \sin\sqrt{\mu}.c$, therefore $c = 0$ and $x = a.\sin\sqrt{\mu}.t$, therefore if $x = a$, $\sqrt{\mu}.t = \frac{\pi}{2}$, or $\frac{5\pi}{2}$, or $\frac{9\pi}{2}$, &c.; and if $x = -a$, $\sqrt{\mu}.t = \frac{3\pi}{2}$, or $\frac{7\pi}{2}$, &c.; consequently the particle passes from the distance $+a$ to $-a$, when $\sqrt{\mu}.t$ increases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, and returns from the distance $-a$ to $+a$, when $\sqrt{\mu}.t$ increases from $\frac{3\pi}{2}$ to $\frac{5\pi}{2}$, and so on for ever; each of these oscillations is therefore per-

formed in a time equal to $\frac{\pi}{\sqrt{\mu}}$, which is the same whether the distance a to which the particle moves be great or small; when $\sqrt{\mu} \, . \, t = 0$, or $= \pi$, or $= 2\pi$, &c., $x = 0$, consequently the particle passes through the origin at the middle of each oscillation, and the velocity at this point is found from the equation $v^2 = \mu\,(a^2 - x^2)$, which becomes when $x = 0$, $v^2 = \mu a^2$, therefore $v = \sqrt{\mu} \, . \, a$, therefore a, the distance to which each oscillation extends, is proportional to the velocity with which the particle passes through the origin.

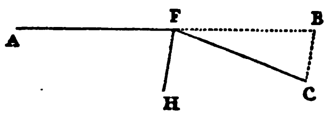
For a third example, let the force be directed (as in the last example) to the origin, and let it vary as the n^{th} power of the distance from that point, then $F = -\mu x^n$, therefore equation (d) becomes $vdv = -\mu x^n dx$, and integrating $\frac{v^2 - v'^2}{2} = \mu \frac{x'^{n+1} - x^{n+1}}{n+1}$, where v' and x' are the velocity and distance at the beginning of the motion; hence v is found, and substituting its value in the equation $dt = \frac{dx}{v}$, the value of t may be found by the integration of this equation.

CHAPTER III.

OF MOTION ALONG A CURVE.

PROP. 1. If a particle moving in the direction AB be acted on by a force in the direction FH, and be thereby caused to move in the direction FC, to find the ratio of the velocities before and after the change.

Draw BC parallel to FH, and cutting the lines AB and FC in B and C. Then the velocity before the change is to that after the change as FB to FC.

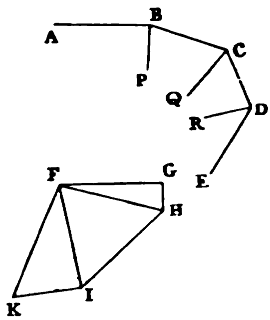


For these velocities are proportional to the forces which would produce them (Ax. 2); but if FB represent the force which would have produced the original velocity, BC will represent the force in the direction FH, which would produce a resultant in the direction FC, and FC will represent that resultant (Part I. Chap. I.); but the velocity after the change is the same as would have been produced by that resultant (Ax. 3); therefore the velocity before the change is to the velocity after the change as FB to FC.

Cor. If the impressed force make equal angles (HFC, HFA) with the two directions of motion, the velocities will be equal, for the angles B and C, being respectively equal to HFA, HFC, will be also equal, therefore FB is equal to FC.

Prop. 2. If a particle moving originally in a given direction AB, be acted on successively by forces in the directions of the right lines BP, CQ, DR, and be thereby caused to move along the lines BC, CD, DE; to find the ratio of the velocities in each direction.

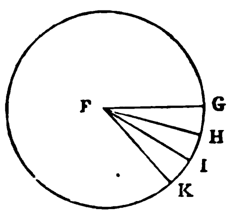
Take any point F, and draw right lines FG, FH, FI, FK, parallel to AB, BC, CD, DE, and from any point G in the first



line, draw GH parallel to the direction of the first force BP, and meeting the second line FH in H, from H draw HI parallel to the second force CQ, and from I draw IK parallel to DR, then will the right lines FG, FH, FI, FK, be in the ratio of the velocities.

For the sides of the triangle FGH are parallel to the sides of the triangle which represent the velocities before and after the action of the force BP (Prop. 1); therefore the velocity in AB is to the velocity in BC as FG to FH; in like manner it may be shewn, that the velocity in BC is to the velocity in CD as FH to FI, and so on to the last.

Prop. 3. If a particle be caused to move in a curved line, by a force always perpendicular to the direction of its motion, its velocity will be unchanged.



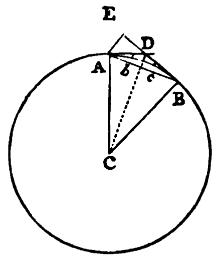
Suppose an indefinite number of lines FG, FH, FI, FK, &c., to be drawn from any point F, parallel to the tangents of the given curve at successive points indefinitely near to one another; and with F as centre, let a circle GHI be described, the tangents to this circle at the points G, H, I, K, &c., are perpendicular to the radii, and therefore (by hypothesis) parallel to the forces at the corresponding points of the given curve; therefore the velocities at these points are proportional to the right lines FG, FH, FI, FK, &c. (Prop. 2); but since these lines are radii of the same circle they are equal, therefore the velocities are also equal.

Scholium. When a particle moves along a *perfectly smooth* curved surface, it is constantly deflected from rectilinear motion by the resistance of this surface, which is a force always

perpendicular to the direction of the motion, therefore in this case the velocity is unchanged.

Prop. 4. If a particle is caused to move in a circle ABC by a force directed always to C the centre of the circle, to find the amount of this force.

Since the force is always perpendicular to the direction of the particle's motion, the velocity is uniform (Prop. 3); let AD and DB be tangents to the circle, they are the directions of the motion at A and B; but the change of direction in the time of passing from A to B, is the same as would have been produced by a single force equal to the resultant of all the forces which do produce the change, let R be this resultant; and since the velocities in the directions AD, DB are equal, the direction of R must make equal angles with these lines (Cor. Prop. 1); therefore if AE be drawn parallel to DC it is parallel to R; and since the three sides of the triangle AED are respectively parallel to the initial motion, the force R, and the resulting motion; R is to the force P which would have produced the velocity in the circle, as AE to AD; but since the three sides of the triangle AED, are respectively perpendicular to the sides of the triangle ABC, AE is to AD as AB to AC, therefore R is to P as AB to AC, and since this is true of any arc, it will be true of the arc A*b*, which may be assumed so small, that the forces directed from it to the centre of the circle, may be considered to be all in the same direction, and therefore their resultant equal to their sum; therefore the sum of all the forces which act on the particle during its motion through this indefinitely small arc A*b* is to P as the chord A*b* to AC, and the same is true of the forces which act on it during its motion through the other small arcs, *bc*, *cd*, &c., of which the entire arc AB is composed; therefore the sum of all the forces during the particle's motion from A to B, is to the force P as the sum of all the



chords Ab, bc, &c. to AC; but since the arcs are assumed indefinitely small, the sum of all their chords is equal to the arc AB; therefore the sum of all the forces which act on the particle during its motion through any arc AB, is to the force P which would have produced the velocity in the circle as that arc AB to the radius AC.

Cor. 1. Since the forces at the different points b, c, d, bear the same ratio to the force P, they are equal; therefore the force directed to the centre of the circle is uniform.

Note. The force with which a revolving body tends to recede from the centre of motion is called *centrifugal force*, and is of course equal and opposite to the force directed to the centre which counteracts it.

Cor. 2. If while the revolving particle moves through an arc equal to radius, another equal particle falls from a state of rest in a straight line towards the centre of the circle by the action of a force equal to that which acts on the revolving particle, its velocity will be equal to that of the revolving particle. For the sum of the forces which act on it is to the force which would produce this velocity, as the arc described by the revolving particle to radius, that is, they are equal.

Cor. 3. The falling particle will in this time pass through a space equal to one-half of the radius, for since it falls by the action of an uniform force, the space passed through is one-half of that passed through in the same time by the revolving particle with the final velocity continued uniform.

Cor. 4. If equal particles revolve in different circles, their centrifugal forces are proportional to the squares of their velocities directly, and to the radii of their orbits inversely.

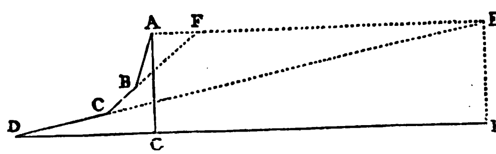
For suppose the particles to fall through spaces equal to the halves of these radii by the action of the same forces, the squares of their velocities will be as the forces multiplied by the spaces fallen through (Chap. II., Prop. 8); therefore the forces are as the squares of the velocities divided by the spaces,

that is, as the squares of the velocities directly, and as the radii inversely.

Cor. 5. Hence if the velocities are equal, the centrifugal forces are inversely as the radii, and if the periodic times are equal, and therefore the velocities proportional to the radii, the centrifugal forces are directly as the radii. Again, if the squares of the periodic times are as the cubes of the radii, and therefore the squares of the angular velocities inversely as the cubes of the radii, the squares of the linear velocities will be inversely as the radii, and therefore, by the preceding corollary, the forces inversely as the squares of the radii.

The latter relation between the distances and periodic times is found to be true of the planets revolving about the sun in orbits nearly circular; therefore if they are retained in their orbits by a force directed to the sun, it must vary inversely as the squares of their distances.

Prop. 5. If a particle fall down a system of inclined planes ABCD, without changing its velocity in passing from one plane to another, its final velocity will be the same as if it had fallen down the vertical height AG.



Through A draw the horizontal line AE, and produce CB, DC, to meet AE in F and E, then the velocity acquired in falling from A to B is the same as would have been acquired by another particle falling from F to B (Chap. II., Cor. Prop. 15), therefore as no velocity is supposed to be lost at B, the two particles will continue to move with equal velocities to C; but the velocity acquired in falling from F to C is the same as would have been acquired by a third particle falling from E to C, therefore these particles will arrive at C with equal velocities, and will move alike from C to D, and so on for any greater number of planes; therefore the final velocity of the particle which falls down ABCD is

equal to that acquired by falling down the inclined plane ED, and therefore equal to that which would have been acquired by falling down its vertical height AG.

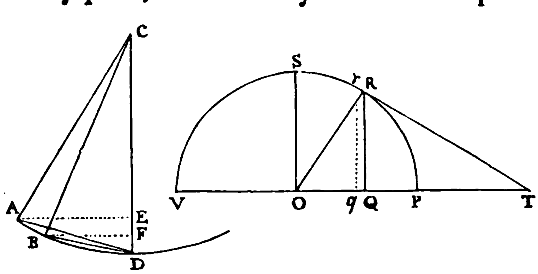
Prop. 6. If a particle fall down any perfectly smooth curved surface, its velocity will be equal to that which would have been acquired by falling down its vertical height.

For if the planes AB, BC, CD (Fig. Prop. 5) be indefinitely diminished in length and increased in number, the system of inclined planes will ultimately become a curved surface, and there will therefore be no loss of velocity in passing from one part of it to another (Sch. Prop. 3); therefore the final velocity will be the same as if it had fallen down the vertical height AG (Prop. 5).

Cor. 1. Hence the square of the velocity is proportional to the height from which the particle has fallen.

Cor. 2. The same is evidently true if the particle be caused to move in a curve by any force always perpendicular to the direction of its motion; as, for example, a pendulous body constrained to describe a circle by a cord attached to a fixed centre.

Prop. 7. To find the ratio of the velocity of a *simple pendulum* (that is, a single particle oscillating in a vertical circle) at any point, to its velocity at the lowest point.



Let A be the position from which the pendulum has been let fall, D the lowest point, and B any other point, C being the centre of the circle; join AD, BD, and with any point O as centre, and a radius OP equal to AD, describe a circle PRS, on OP take a part OQ equal to BD, and erect a perpendicular QR, the velocity at B is to the velocity at D as QR to OP.

For draw AE and BF perpendicular to CD, and therefore horizontal, and join OR, the square of the velocity at B is to the square of the velocity at D as EF to ED (Prop. 6, Cor. 1); but FD is to DE as the square of BD is to the square of AD, or as the square of OQ to the square of OR; therefore EF is to ED as the square of QR to the square of OR, and therefore the square of the velocity at B is to the square of the velocity at D, as the square of QR to the square of OR, and the velocities themselves are as the lines QR, OR.

Prop. 8. To find approximately the time of falling from A to D, the arc AD being so small, that it may without sensible error be considered equal to its chord.

Let the construction of the last proposition remain, and at R draw the tangent RT, and while the pendulum B moves from A to D, let the perpendicular RQ move from P to OS, the velocity of the point Q being always equal to that of the pendulum at the same instant; then since OP is supposed equal to the arc AD, the pendulum will move from A to D in the same time that the perpendicular moves from P to OS; and OQ being also equal to DB, the pendulum will be at B when the perpendicular is at QR, draw *qr* parallel and indefinitely near to QR, the lines Q*q* and R*r* will be described in the same time by the points Q and R, therefore the velocity of Q is to the velocity of R as Q*q* to R*r*, that is, as QT to RT, or as QR to OR; but the velocity of the pendulum at B is to its velocity at D as QR to OR (Prop. 7), therefore the velocity at B is to the velocity at D as the velocity of Q to the velocity of R; but the velocity of the pendulum at B is equal to the velocity of Q, therefore its velocity at D is equal to the velocity of R, consequently the point R moves uniformly from P to S with the same velocity that the pendulum has at the lowest point D; and the time of falling from A to D is equal to the time of moving with this velocity from P to S.

It is evident that the entire time of oscillation is twice the time of falling from A to D.

Prop. 9. The time of oscillation of a pendulum is to the time of falling down a height equal to half the length DC of the pendulum, as the circumference of a circle to its diameter.

For the velocity at the lowest point D is equal to the velocity which would have been acquired by falling down the chord AD (Prop. 6), and the time of moving along that chord with the velocity at the lowest point, is one-half the time of falling down the chord (Chap. II., Prop. 3), that is, one-half the time of falling down the diameter of the circle (Cor. 1, Prop. 14), or equal to the time of falling down one-fourth of the diameter, which is one-half the length of the pendulum; but the time of an oscillation is equal to the time of moving along the semicircle VSP, with the velocity of the pendulum at D (Prop. 8); therefore it is to the time of moving along the chord AD with the same velocity, as VSP to AD, or to OP which is equal to AD, or as 2 VSP (the circumference) to twice OP (the diameter); and it was before proved, that the time of moving along the chord AD with that velocity, is equal to the time of falling down a height equal to half the length of the pendulum, therefore the time of the oscillation is to the time of falling down that height as the circumference of a circle to its diameter.

Cor. 1. Hence the time of falling down an indefinitely small arc of a circle, terminated at its lowest point, is to the time of falling down the chord of that arc as the circumference of a circle to four times its diameter; for the time of falling down the arc is half the time of an oscillation; and the time of falling down the chord is equal to the time of falling down the diameter, that is, twice the time of falling down half the length of the pendulum: but it is to be observed, that this inequality of the times of falling down the arc and its chord, does not at all arise from the difference of the two lines in length (for they were supposed small enough to be looked on as equal without sensible error); but from the fact, that the accelerating force at the beginning of the motion is nearly

double as great in the arc as in its chord, on account of the inclination of the former to the horizon being exactly double of the inclination of the latter.

Cor. 2. The time of oscillation of a pendulum depends only on its length, and is independent of the length of the arc described, provided it be not so great as to differ perceptibly from its chord.

Cor. 3. In different pendulums the time of oscillation is proportional to the square root of the length; for the time of falling down the half-length of the pendulum varies in this ratio.

Note. From this proposition the time of falling down any given height may be found with great accuracy, for the duration of each oscillation can be ascertained very exactly by observing the time of performing a great number of oscillations, and dividing this time by the number, the quotient will give the time of one oscillation; and if there be any error in the observation of the entire time, this error divided by the number of oscillations will be the error in each, which will thus be greatly diminished; but the time of each oscillation thus accurately found, is to the time of falling down half the length of the pendulum, as the circumference of a circle to its diameter, that is to say, in a known ratio, and therefore the time of falling down this height is also known; and that time is to the time of falling down any other height in the subduplicate ratio of the heights, whence the time of falling down any height can be computed. Also the final velocity may be immediately found from the same observation, for it is the velocity with which the same space would be passed through in half the time. Now this velocity, being produced in a known time by the action of the force of gravity, is a measure of that force; whence it appears, that the observation of the oscillations of a pendulum furnishes an accurate measure of the force of gravity, by means of which it has been computed that this force is capable of producing in

one second a velocity of nearly thirty-two feet two inches per second, but that it is not precisely the same at all parts of the earth's surface: the exact variation may be found by the following proposition.

Prop. 10. If the force of gravity vary in different parts of the earth, the length of the pendulum which oscillates in one second varies in the same ratio.

For the times of oscillation being equal in the two places, the times of falling down heights equal to the halves of the lengths of the pendulums are also equal, that is, the times of describing spaces equal to the lengths of the pendulums with the final velocities continued uniform, therefore the lengths are proportional to the final velocities; but the times of acquiring these velocities being equal, the velocities are as the forces, therefore the forces are proportional to the lengths.

It is found by experiment, that the length of the second's pendulum increases as we travel from the Equator towards either Pole; the increase in the length being proportional to the square of the sine of the latitude, or, which is the same thing, the decrease in the length as we proceed in the opposite direction from the Pole to the Equator being proportional to the square of the cosine of the latitude. And by the last proposition the force of gravity is diminished in the same proportion. This variation in the force of gravity is partly caused by centrifugal force; for if the earth were a sphere, since all its parts revolve in the same time about its axis, the centrifugal force at each point is proportional to the radius of the circle which it describes (Prop. 4, Cor. 5), that is, proportional to the cosine of the latitude; but this force acting in a direction parallel to the Equator may be resolved into two, one horizontal (which has, in fact, the effect of altering the figure of the earth from a sphere to an oblate spheroid), the other vertical, which diminishes the force of gravity, and this part is to the whole centrifugal force, as the cosine of the latitude to radius, therefore it is to the centrifugal force at the

Equator as the square of the cosine of the latitude to the square of radius; and the diminution of the force of gravity resulting from this cause is proportional to the square of the cosine of latitude. If the circumference of the Equator be 25,000 miles, and the space fallen through in one second by the action of gravity sixteen feet one inch, it may be easily computed, that this diminution at the Equator is to the whole force of gravity as 1 to 289 nearly; the actual diminution is greater than this, on account of the oblate form of the earth.

Note to Chapter III.

It follows from Prop. 3, that the motion of a particle on a curve will follow the same law as its motion in a right line, provided that the accelerating force is the same; now in the simple pendulum, the accelerating force is $g \sin \theta$, where θ is the angle that the tangent to its path makes with the horizon, or, which is the same thing, the angle which the pendulum makes with the vertical: but if l be the length of the pendulum, and s its distance from the lowest point, $l\theta = s$, therefore the accelerating force is $g \sin \frac{s}{l} = g\left(\frac{s}{l} - \frac{s^3}{1.2.3l^3} + \&c.\right)$; if s be so small that its cube may be neglected, this force is proportional to s, therefore the time of an oscillation is constant (Note, Chap. II.); and equal to $\pi \sqrt{\frac{l}{g}}$, the same as that found in Prop. 8; for a nearer approximation, let the two first terms of the above development be taken as the value of the accelerating force, the fifth and higher powers of s being neglected, then $vdv = g\left(\frac{sds}{l} - \frac{s^3ds}{6l^3}\right)$, and integrating

$$v^2 = g\left(\frac{s'^2 - s^2}{l} - \frac{s'^4 - s^4}{12l^3}\right), \text{ and T (the time of an oscillation)} =$$

$$\sqrt{\frac{l}{g}}\int\left(1 - \frac{s'^2 + s^2}{12l^2}\right)^{-\frac{1}{2}} \frac{ds}{\sqrt{(s'^2 - s^2)}} = \sqrt{\frac{l}{g}}\int\left(1 + \frac{s'^2 + s^2}{24l^2}\right) \cdot \frac{ds}{\sqrt{(s'^2 - s^2)}} = \pi\sqrt{\frac{l}{g}}\left(1 + \frac{s'^2}{16l^2}\right),$$

therefore the time of oscillation of a pendulum through the arc 2θ is nearly proportional to $1 + \frac{1}{16}\sin^2\theta$.

If it be required to find a curve in which the oscillations are perfectly

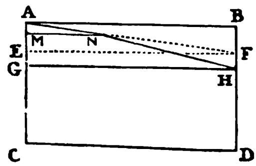
isochronous, let θ be the angle which a tangent to this curve makes with the horizon, then $g \sin\theta$ is the accelerating force, and this should be proportional to the distance from the lowest point; let this distance be s, therefore $a \sin\theta = s$; but if the curve be referred to rectangular coordinates, the axis OX being horizontal and OY vertical, $\sin\theta = \frac{dy}{ds}$, therefore $ady = sds$, $ay = \frac{1}{2}s^2 + c$; or (if the origin be taken at the lowest point, so that $c = 0$) $ay = \frac{1}{2}s^2$; an equation which may easily be identified with that of a cycloid, whose axis is vertical and equal to $\frac{1}{2}a$.

If the motion in this curve were accelerated or retarded by a constant force, as friction, the action of clock-work, &c., it is manifest that this will only have the effect of causing the entire accelerating force to be proportional to the distance from a point on the cycloid different from its vertex, and thus will increase or diminish the amplitude of the oscillations, without producing any effect on their duration; the same is nearly true of the resistance of the air, although it is not a constant force, but depending on the velocity.

CHAPTER IV.

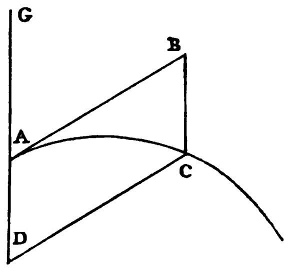
OF PROJECTILES.

PROP. 1. If there be two equal particles, one at rest at A, the other moving with uniform velocity from A towards B, and if each be acted on by a succession of equal and simultaneous impulses parallel to BD, which cause the first particle to move from A to C in the time in which the second would have moved with its uniform velocity from A to B, then the second particle will move in the same time from A to the opposite angle D of the parallelogram ABCD.



For let F be the force capable of producing the uniform velocity of the second particle, and let the first of the impulses be to F as AE to AB, complete the parallelogram ABFE, and AF will be the resultant of the first impulse and of the force F, therefore the second particle would move uniformly from A to F, while the first moves from A to E (Ax. 3). Now if the second impulse acts on the two particles, while one is at M, and the other at N on the line AF, then since AM is to AN as the velocities of the two particles, MN is parallel to EF, and if this second impulse would cause each particle to move through a space equal to FH in the remainder of the time, then it can be shewn in the same manner, that the second particle will move from N to H in this time, and if EG be equal to FH, the first particle will move to G in the same time; but since EG and FH are equal, GH is parallel to EF, and therefore to AB; in like manner, if there be any number of impulses which bring the first particle in the given time to any point C, they will bring the second particle in the same time to D, the right line CD being parallel to AB.

Prop. 2. If a particle be projected in the direction AB with a velocity sufficient to bring it from A to B in a given time t, and if AD be the space through which it would fall by the force of gravity in the same time; complete the parallelogram ABCD, then C will be the actual place of the body at the end of the time t.



For let an equal particle fall from A at the same time, then the two particles being acted on by gravity, that is, by a succession of equal and simultaneous forces parallel to BC at indefinitely short intervals, which cause one particle to move from A to D in the same time in which the other would have moved with its original velocity from A to B, this particle will move from A to C in the same time (Prop. 1).

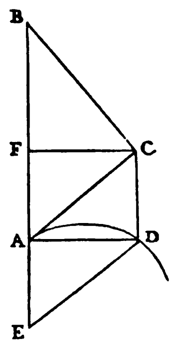
Prop. 3. If a particle be projected in the direction AB with any given velocity, and if AG be the height due to that velocity, and if the right line CD be drawn from any point C on the path described by the particle, parallel to AB, and meeting the vertical through the point of projection in D, the square of CD is equal to four times the rectangle under AG and AD.

Complete the parallelogram ABCD, then the time of falling from A to D is equal to the time of moving with the velocity of projection from A to B, this being the time in which the projectile moves from A to C (Prop. 2); but the time of falling from G to A is equal to the time of moving with the same velocity through a space equal to 2AG (Chap. II., Prop. 5), therefore the time of falling through AD is to the time of falling through GA as AB to 2AG, and therefore the squares of these times are as the square of AB to four times the square of AG; but the spaces fallen through are as the squares of the times of falling (Chap. II., Prop. 7), therefore AD is to AG as the square of AB to four times the square of AG, and therefore

the square of AB (or of CD which is equal to AB) is equal to four times the rectangle under AG and AD.

Note. The curve AC is that called by geometers a parabola, AD is one of its diameters, and AB a tangent; a right line drawn parallel to the tangent, as CD, is called an ordinate, and the intercept AD its absciss; and since by the nature of the parabola, the square of any ordinate is equal to the rectangle under the absciss and the parameter, it follows that four times AG is the parameter of the diameter AD, therefore if a parabola be described touching AB in A, and having AD for a diameter and 4 AG for its parameter, this parabola will be the path of the projectile.

Prop. 4. To find where a projectile will strike the horizontal plane passing through the point of projection.



Let AB be four times the height due to the velocity of projection, from B let fall BC perpendicular on AC the direction of projection, and from C let fall CD perpendicular on the horizontal plane AD, then D is the point where it will strike this plane.

For draw DE parallel to AC, and meeting the vertical AE in E, then in the triangles ABC, EAD, the angles ACB and EAD are equal, being right angles, and BAC is also equal to AED, therefore the triangles are similar, and BA is to AC (or DE) as DE to AE, therefore the square of DE is equal to the rectangle under AB and AE, that is, equal to four times the rectangle under AE and the height due to the velocity of projection; therefore D is a point on the path of the projectile (Prop. 3), and it is also on the horizontal plane passing through A, therefore it is the required point.

Cor. 1. The distance AD is called the horizontal range of the projectile, and it can never be greater than twice the height due to the velocity of projection, for if CF be drawn parallel to AD, it is a mean proportional between AF and FB,

therefore it cannot be greater than the half of AB; but AD is equal to CF, and AB is four times the height due to the velocity of projection, therefore AD is not greater than twice that height.

Cor. 2. If the velocity of projection be given, the horizontal range is a maximum when the angle CAD of projection is half a right angle.

For in this case the triangle ACB is isosceles, and CF is equal to one-half of AB, therefore AD the horizontal range is equal to twice the height due to the velocity; but by the preceding corollary, it cannot be greater than this.

Cor. 3. It is evident that any horizontal range less than the maximum may be attained by two different angles of projection, which are complementary to one another.

Note to Chapter IV.

The theory of projectiles may be easily derived from the equations (1) in the note to Chap. II.; for if the point of projection be assumed as the origin of rectangular coordinates, the axis OX being vertical, and the plane of XY containing the direction of projection, these equations will become

$$\frac{da}{dt}=-g, \quad \frac{db}{dt}=0, \quad \frac{dc}{dt}=0;$$

and integrating $a-a'=-gt$, $b-b'=0$, $c-c'=0$, where a', b', c' are the components of the velocity of projection; let v' be this velocity, and θ the angle which its direction makes with OY, then $a'=v'\sin\theta$, $b'=v'\cos\theta$, $c'=0$; and substituting these values, and putting for a, b, c their values in equations (2),

$$\frac{dx}{dt}=v'\sin\theta-gt, \quad \frac{dy}{dt}=v'\cos\theta, \quad \frac{dz}{dt}=0;$$

and integrating $x=v'\sin\theta t-\frac{gt^2}{2}$, $y=v'\cos\theta t$, $z=0$ (no constants are added, because that at the beginning of the motion $x=0$, $y=0$, $z=0$); hence it appears that the path of the projectile is in the plane of xy, and its equation may be found, by eliminating t, to be

$$y^2-\frac{v'^2\sin 2\theta}{g}y+\frac{2v'^2\cos^2\theta}{g}x=0,$$

which is evidently the equation of a parabola, the coordinates of whose

vertex are $y' = \frac{v'^2}{2g}$ sin 2θ, $x' = \frac{v'^2}{2g}$ sin² θ, and its principal parameter $p = \frac{2v'^2 \cos^2\theta}{g}$; or if h be the height due to the velocity, $y' = h \sin 2\theta$, $x' = h \sin^2\theta$, $p = 4h \cos^2\theta$, whence the equation of the directrix $x = x' + \frac{1}{4}p$, becomes $x = h$, and the coordinates of the focus are $y = y' = h \sin 2\theta$, $x = x' - \frac{1}{4}p = -h \cos 2\theta$; also the horizontal range $= 2y' = 2h \sin 2\theta$.

For another example of the application of these equations, let the force F be directed to a fixed point, and let the particle be projected in any direction not passing through this point, then the fixed point may be assumed as the origin of coordinates, and the plane of xy be that which contains this point and the direction of projection, then $X = F\frac{x}{r}$, $Y = F\frac{y}{r}$, $Z = F\frac{z}{r}$, and substituting these values in equations (3),

$$\frac{d^2x}{dt^2} = F\frac{x}{r},\quad \frac{d^2y}{dt^2} = F\frac{y}{r},\quad \frac{d^2z}{dt^2} = F\frac{z}{r}; \tag{a}$$

multiplying the two last of these equations by z and y respectively, and subtracting, $\frac{yd^2z - zd^2y}{dt^2} = 0$, and integrating $\frac{ydz - zdy}{dt} = c$; but since at the beginning of the motion $z = 0$, $\frac{dz}{dt} = 0$, therefore $c = 0$, and $\frac{ydz - zdy}{dt} = 0$; dividing by y^2, and integrating again, $\frac{z}{y} = c' = 0$, therefore the path of the particle is in the plane of xy; again, multiplying the two first of equations (a) by y and x respectively, subtracting and integrating, $\frac{xdy - ydx}{dt} = 2h$, where h is a constant depending on the circumstances of projection; to render this equation more easily integrable, let r be the distance of the particle from the origin of coordinates, and ω the angle contained by r and OX, then $\frac{y}{x} = \tan\omega$, and differentiating $\frac{xdy - ydx}{x^2} = \sec^2\omega d\omega$, but $xdy - ydx = 2hdt$, and $x^2 \sec^2\omega = r^2$, therefore

$$\frac{r^2 d\omega}{2} = hdt; \tag{b}$$

the first member of this equation is the area described by the radius vector r as it moves through the angle $d\omega$, therefore this area is proportional to the time of describing it. Again, multiplying the two first of the equations (a) by $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and adding, $\frac{dxd^2x + dyd^2y}{dt^3} = F\frac{xdx + ydy}{rdt} = F\frac{dr}{dt}$, and integrating

$$\frac{dx^2 + dy^2}{dt^2} = 2\int F dr + k, \tag{c}$$

the first member of this equation is the square of the velocity, and is evi-

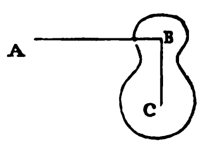
dently equal to $\frac{r^2 d\omega^2 + dr^2}{dt^2}$, and substituting for dt its value derived from equation (b), $2h^2 \frac{r^2 d\omega^2 + dr^2}{r^4 d\omega^2} = 2\int \mathrm{F}dr + k$, which is the polar equation of the orbit of the projectile; its second member cannot be integrated unless F be a given function of r, and even then the integration is very difficult, except in some particular cases. For example, let $\mathrm{F} = \frac{-\mu}{r^2}$, then $\int \mathrm{F}dr = \frac{\mu}{r}$, and if for r we substitute $\frac{1}{u}$, the equation becomes $2h^2 \frac{du^2}{d\omega^2} = \mu u - 2h^2u^2 + k$, therefore $d\omega = \frac{2h du}{\sqrt{(2k - 4h^2u^2 + 2\mu u)}}$, and by integration, $\omega - a = \cos^{-1}\frac{4h^2u - \mu}{\sqrt{(8h^2k + \mu^2)}}$, or $u = \frac{\mu}{4h^2} + \frac{1}{4h^2}\sqrt{(8h^2k + \mu^2)} \cos(\omega - a)$; the equation of a conic section, of which the origin is focus, $\frac{4h^2}{\mu}$ the semiparameter, and $\sqrt{\left(1 + \frac{8h^2k}{\mu^2}\right)}$ the eccentricity. The constants h and k may be determined by means of the original conditions of the motion, for if v be the original velocity, and l the original distance from the fixed point, and θ the angle contained by their directions, it is easy to see that $\mathrm{v}dt . l \sin\theta = r^2 d\omega = 2hdt$ by equation (b), therefore $h = \frac{1}{2}\mathrm{v}l\sin\theta$; also, by equation (c), $\mathrm{v}^2 = 2\left(\frac{\mu}{l} + k\right)$, therefore $k = \frac{1}{2}\mathrm{v}^2 - \frac{\mu}{l}$. Now the conic section will be an ellipse, hyperbola, or parabola, according as the eccentricity is less than, greater than, or equal to unity, that is according as k is negative, positive, or cypher; therefore it will be an ellipse if $\mathrm{v}^2 < \frac{2\mu}{l}$, a parabola if $\mathrm{v}^2 = \frac{2\mu}{l}$, or an hyperbola if $\mathrm{v}^2 > \frac{2\mu}{l}$; in the first case, the axis major of the ellipse may be found from the above values of the semiparameter and the eccentricity, it will be $\frac{-\mu}{k}$, or (substituting the value already found for k) $\frac{2\mu l}{2\mu - \mathrm{v}^2 l}$, a value depending wholly on v and l, and independent of θ. Again, if the eccentricity were cypher, $h^2k = -\frac{\mu^2}{8}$, that is, $\mathrm{v}^4 - 2\frac{\mu}{l}\mathrm{v}^2 + \frac{\mu^2}{l^2\sin^2\theta} = 0$, and in order that this may give a possible value for v the original velocity, $\sin\theta$ must be $= 1$, therefore $\mathrm{v}^2 = \frac{\mu}{l} = \frac{1}{2}$ the square of the velocity in the parabola, consequently if the particle be projected with this velocity at an angle $\theta = 90°$, it will describe a circle. It is evident that this is the least value of h^2k, which will give a possible value of v, therefore no original velocity can make the above value of the eccentricity imaginary. To find the periodic time in the ellipse, we have

$T = \frac{\text{area of ellipse}}{h}$, that is, (if a and b be the semiaxes) $= \frac{\pi ab}{h}$, but the semi-parameter $\frac{4h^2}{\mu} = \frac{b^2}{a}$, therefore $\frac{b}{h} = 2\sqrt{\frac{a}{\mu}}$ and $\frac{\pi ab}{h} = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$, therefore the squares of the periodic times are as the cubes of the major axes of the orbits. Now it has been discovered by Kepler—first, that the planets move round the sun in ellipses, and that the centre of the sun is one focus of each of these ellipses; secondly, that they describe equal areas about this point in equal times; and thirdly, that the squares of the periodic times are as the cubes of the major axes; whence it appears, that the force which retains the planets in their orbits is directed to the centre of the sun, and inversely proportional to the square of the distance; for it may easily be proved, that these conditions of their motion could not be produced by any other force; in fact, the second law proves that the force is directed to the centre of the sun; the first law determines the law of the variation of this force from one point to another in the same orbit; and the third law proves, that the force acting on one planet is to that acting on any other inversely as the squares of their distances from the sun. For a fuller account of the motions of bodies acted on by central forces, see Luby's Physical Astronomy, and Airy's Tracts on the Lunar and Planetary Theories.

CHAPTER V.

OF THE ROTATION OF RIGID BODIES.

Prop. 1. If a rigid body containing a fixed axis be acted on by a given force in a plane perpendicular to that axis, to determine its motion.



Let the fixed axis be perpendicular to the plane of the figure at C, let AB be the direction of the force, and let it be sufficient to produce a given velocity V in a given particle M, if that particle were detached from the body. Now since the axis is fixed, the particles of which the body is composed can only move in circles whose centres are on this axis, and in planes perpendicular to it, and since all these circles are described in the same time, the velocities of the particles are proportional to their distances from the axis, that is, if the particles m, m', m'', &c., be at distances r, r', r'', &c., from the axis, their velocities are ωr, $\omega r'$, $\omega r''$, &c., where ω is a quantity depending on the velocity of rotation about the axis; and the forces which would produce these velocities are to the force AB as $m\omega r$, $m'\omega r'$, $m''\omega r''$, &c., to MV; therefore their moments are to the moment of AB as $m\omega r^2$, $m'\omega r'^2$, $m''\omega r''^2$, &c., to MV $\times$ BC; but since there is equilibrium between these forces and a force equal and opposite to AB (Ax. 5, *note*) the sum of their moments about the fixed axis is equal to the moment of AB about the same axis, that is to say, $m\omega r^2 + m'\omega r'^2 + m''\omega r''^2 + \&c. = \text{MV}.\text{BC}$, therefore $\omega = \dfrac{\text{MV}.\text{BC}}{mr^2 + m'r'^2 + m''r''^2 + \&c.}$, the denominator of this fraction being the sum of each particle multiplied by the square of its distance from the axis; ω being thus found, the velocity

of each particle is at once determined, for it is equal to ω multiplied by its distance from the axis.

Thus, if the body consisted of two particles m, m' at distances r, r' from the axis, and if the force was applied to one of these particles m, in a direction perpendicular to the plane containing r and the fixed axis, and were sufficient to produce in it, if it were free, the velocity v, then $\omega = \frac{m\text{v}r}{mr^2 + m'r'^2}$, and the actual velocity of m will be $\omega r = \text{v}\,\frac{mr^2}{mr^2 + m'r'^2}$; if the two particles were equal, and at equal distances, then $mr^2 = m'r'^2$, and the velocity of m will be $\frac{1}{2}$v, if they were equal and $r' = 2r$, then $m'r'^2 = 4mr^2$, and the velocity of m will be $\frac{1}{5}$v, and in like manner the velocity may be found in any other case.

Cor. It is evident that if the body were in motion before the force acted on it, the change of velocity of each particle might be found in the same way.

Note.—The angle described being measured by the arc divided by the radius of the circle, it follows that the angular velocity is measured by the linear velocity divided by the radius, therefore ω is the measure of the angular velocity; also the quantity $mr^2 + m'r'^2 + m''r''^2 +$ &c., being the measure of the moment of the force which will produce a given angular velocity, is called the *moment of inertia* of the body with respect to that axis, its effect on the angular velocity being analogous to the effect of inertia on linear velocity.

Prop. 2. The moment of inertia of a body with respect to any fixed axis, exceeds its moment of inertia with respect to a parallel axis passing through the centre of gravity, by the product of the quantity of matter in the body multiplied by the square of the distance between the two axes.

Let m be one of the particles of which the body is composed, and let the plane mAG passing through m cut the two axes perpendicularly at A and G, join mA, mG, and AG, and

as AG to AO, from which proportion AO may be found if the moment of inertia be known, for

$$\text{AO} = \frac{\text{the moment of inertia with respect to A}}{\text{M} \times \text{AG}},$$

and O is the required point.

Cor. 1. The point O is called the centre of oscillation, its position will evidently be altered by changing the axis of rotation A, and is determined by this equation.

Cor. 2. The small oscillations of any body about a fixed horizontal axis, which does not pass through the centre of gravity, are nearly isochronous, for they are the same as the oscillations of the whole mass of the body collected at the point O; but this would be a simple pendulum, whose small oscillations have been already proved to be nearly isochronous. (Chap. III., Prop. 9).

Cor. 3. The time of an oscillation is the same in any two pendulums, whether simple or compound, if the lengths AO be equal, and if the velocity V, which the force of gravity would produce in the two bodies, be the same, and hence is derived a most accurate test for determining whether gravity acts equally on all bodies, by which the truth of this property is fully established.

Prop. 4. Given the centre of oscillation O of a body oscillating about a given axis A, to find the moment of inertia with respect to a parallel axis passing through the centre of gravity.

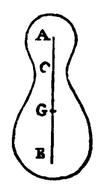
Let the plane of the figure pass through the point O, and cut the axis perpendicularly at A, then the centre of gravity must be in the right line AO (Prop. 3), let G be the centre of gravity, and let M be the quantity of matter in the body, then AO is equal to the moment of inertia with respect to A divided by M $\times$ AG (Cor. 1, Prop. 3), therefore the moment of inertia with respect to A, is equal to M multiplied by the rectangle under AG and AO; but the moment of inertia with respect to A exceeds the moment of inertia with respect to the parallel

axis through G, by M multiplied by the square of AG (Prop. 2), therefore the moment of inertia about this axis through G is equal to the excess of M multiplied by the rectangle under AG and AO, over M multiplied by the square of AO, that is, equal to M multiplied by the rectangle under AG and GO.

Cor. 1. The centre of oscillation is always on the opposite side of the centre of gravity from the fixed axis, for since $M \times AG \times AO$ exceeds $M \times AG^2$, AO must be greater than AG.

Cor. 2. If the body be suspended on any other axis parallel to the given one, the rectangle under the distances of this axis, and the corresponding centre of oscillation, from the centre of gravity, is equal to the rectangle under AG and GO, for each rectangle is equal to the moment of inertia with respect to the parallel axis through the centre of gravity divided by the mass of the body.

Prop. 5. If a body contain two parallel axes, one of which passes through the centre of oscillation of the other, then the latter axis will also pass through the centre of oscillation of the former, and the time of oscillation about one axis will be equal to the time of oscillation about the other.



Let the plane of the figure pass through the centre of gravity G, and cut the two axes perpendicularly in A and B, then since the centre of oscillation corresponding to A is on the axis B by hypothesis, and is also on the right line AG (Prop. 3), it is at B, the intersection of these lines. If it be possible, let A not be the centre of oscillation corresponding to the axis B, but let it be some other point C, then C must be on the right line BG (Prop. 3), and at the opposite side of G from B (Prop. 4, Cor. 1), therefore on the same side with A, and therefore GC is not equal to GA; but the rectangle under BG and GC is equal to the rectangle under BG and GA (Prop. 4, Cor. 2), therefore GC is equal to GA, which is absurd, therefore the centre of oscillation corresponding to B is on the axis A. But the time of oscillation about either axis is

equal to the time of oscillation of a simple pendulum whose length is equal to AB, therefore the times are equal to one another.

Cor. 1. It is evident that the body will perform its oscillations in equal times about any other axis parallel to A and B, whose distance from the centre of gravity is equal either to AG or to BG.

Cor. 2. If the distances AG and BG are equal, the time of oscillation about either of these axes will be less than about any other axis parallel to them; for since the rectangle under AG and BG is given, their sum (which is the length of the corresponding simple pendulum) is a *minimum* when they are equal (see Euclid, B. 2, Prop. 5).

Cor. 3. If the distances AG and BG are unequal, there are four parallel axes cutting the right line AB, about which the body will oscillate in the same time, namely, two at each side of the centre of gravity, whose distances from that point are equal respectively to AG and BG.

Prop. 6. To find by experiment the centre of oscillation of a pendulum oscillating about a given axis.

Find the centre of gravity of the body, and through it draw a right line perpendicular to the given axis, and suspend the pendulum on several different axes, cutting this right line at the opposite side of the centre of gravity, and find by trial if the oscillations about any of these axes (whose distance from the centre of gravity is not equal to the distance of the given axis), are performed in times equal to the time of oscillation about the given axis; if so, the point where this axis cuts the perpendicular is the centre of oscillation; and if there be no such point, then the distance of the centre of oscillation is equal to the distance of the axis from the centre of gravity.

For there can be but two points on the same side of the centre of gravity, through which, if axes be drawn parallel to the given one, the oscillations will be performed in equal times, one of them is at a distance from the centre of gravity

equal to the distance of the given axis, the other point is the centre of oscillation (Prop. 5, Cor. 3); and when the distance of this point from the centre of gravity is equal to the distance of the given axis, the two points coincide at the centre of oscillation, as appears from Prop. 5, Cor. 2.

N. B.—It is necessary that the axes should be parallel to one another, as the distance of the centre of oscillation from the centre of gravity depends on the moment of inertia with respect to the parallel axis passing through the centre of gravity, but this is in general altered by any change in the direction of that axis; in practice, this method of finding the centre of oscillation is more conveniently effected by fixing an axis parallel to the given one at a point as near as possible to the centre of oscillation, and then by means of weights which are capable of being moved along the length of the pendulum, to alter the position of the centre of oscillation, until it comes to coincide accurately with the second axis, which will be ascertained by the equality of the times of oscillation.

Note to Chapter V.

The moment of inertia of the circumference of a circle, with respect to an axis passing through its centre and perpendicular to its plane, is evidently equal to the circumference multiplied by the square of the radius, that is, if r be the radius, $2\pi r^3$. Also, since the square of the distance of any point from the centre is equal to the sum of the squares of its distances from two perpendicular diameters, it follows that this moment of inertia is equal to the sum of the moments of inertia with respect to these two diameters, or to twice the moment of inertia with respect to one of them, therefore the moment of inertia of the circumference of a circle with respect to its diameter is πr^3.

Again, since a circular plate may be conceived to consist of an infinite number of concentric circles, its moment of inertia is equal to the sum of the moments of inertia of all these circles, that is, if the axis pass through the centre, the moment of inertia is equal to $\int 2\pi r^3 dr$, or to $\int \pi r^3 dr$, according as the axis is perpendicular to, or in the plane of the circle, the former integral is $\frac{1}{2}\pi r^4$, the latter $\frac{1}{4}\pi r^4$, r being the radius of the circular plate.

Hence the moment of inertia of a solid of revolution, with respect to any

axis, may be easily found. First, let the axis be the axis of revolution, and let y be the ordinate of the generating curve perpendicular to this axis, and x the abscissa; then since the moment of inertia of the solid is the sum of the moments of inertia of all the circular plates perpendicular to the axis of which it is composed, the moment of inertia with respect to the axis of revolution is evidently $\frac{1}{2}\pi\int y^4dx$.

Next, let the axis be perpendicular to the axis of revolution, then the moment of inertia of each circular plate, with respect to this axis, exceeds its moment of inertia with respect to a diameter of the plate by the quantity of matter in the plate multiplied by the square of its distance from the axis; therefore if this distance be x it is equal to $\frac{1}{4}\pi y^4dx + \pi y^2x^2dx$, and the moment of inertia of the entire solid is $\frac{1}{4}\pi\int y^4dx + \pi\int y^2x^2dx$.

Thirdly, if the axis cut the axis of revolution at any angle θ, let the point of intersection be the origin, and the axis of revolution the axis of x, as before, and let the axis of y lie in the plane of the angle θ, the coordinates being rectangular, then the square of the distance of any point x, y, z from the proposed axis is equal to $x^2\sin^2\theta - 2xy\sin\theta\cos\theta + y^2\cos^2\theta + z^2$, and the moment of inertia of the particle dm situated at this point is equal to the product of dm multiplied by the square of its distance, therefore the moment of inertia of the whole body is $\sin^2\theta\int x^2dm - 2\sin\theta\cos\theta\int xydm + \cos^2\theta\int y^2dm + \int z^2dm$; but since the axis of x passes through the centre of gravity of each circular section of the body, it is evident that $\int xydm = 0$, and therefore the moment of inertia is reduced to the three remaining terms; but it is evident also that $\int y^2dm = \int z^2dm = \frac{1}{4}\pi\int y^4dx$ (since it is half the moment of inertia with respect to the axis of revolution), and that $\int x^2dm + \int y^2dm = \frac{1}{4}\pi\int y^4dx + \pi\int y^2x^2dx$ (since it is the moment of inertia with respect to the axis of z), therefore, substituting for $\int y^2dm$ its value, we have $\int x^2dm = \pi\int y^2x^2dx$, and the expression for the moment of inertia with respect to the proposed axis becomes

$$\pi\sin^2\theta\int y^2x^2dx + \frac{\pi}{4}(\cos^2\theta + 1)\int y^4dx.$$

It is easy to see that $\int xzdm$ and $\int yzdm$ are equal to cypher as well as $\int xydm$. Axes of coordinates which possess this property are called principal axes of the body, and the discovery of such axes simplifies very much the theory of rotation, but the necessary calculations are too long to be introduced here.

For an application of the preceding equations, let it be proposed to find the moment of inertia of a cylinder of revolution with respect to an axis passing through the centre of its base, the radius of this base being r, and the length of the cylinder l; in this case $y = r$, therefore $\int y^4dx = r^4l$, and $\int y^2x^2dx = \frac{1}{3}r^2l^3$; consequently the moment of inertia with respect to any

axis passing through the origin, and making an angle θ with the axis of revolution, is $\frac{1}{3}\pi \sin^2\theta r^2 l^3 + \frac{1}{4}\pi(1 + \cos^2\theta)r^4 l$.

In like manner, the moment of inertia of a right cone with respect to any axis through its vertex is $\frac{1}{5}\pi a^2 h^5 \sin^2\theta + \frac{1}{20}\pi a^4 h^5(1 + \cos^2\theta)$, h being the altitude of the cone, and a the tangent of half the vertical angle.

The moment of inertia of a sphere with respect to its diameter, is found by substituting for y^2 its value $r^2 - x^2$, r being the radius of the sphere, and the origin of coordinates being at its centre; thus the expression for the moment of inertia becomes

$$\frac{\pi}{2}\int_{-r}^{+r}(r^2 - x^2)^2 dx = {}_{15}\pi r^5.$$

The moment of inertia being found as above for all axes which cut the axis of revolution, it is easy to find the same for any other axis by means of Prop. 2.

The general problem to find the moment of inertia of a body of any given form, with respect to a given axis oz, requires three integrations of the quantity $(x^2 + y^2)dm$, with respect to x, y, and z successively; if the body be homogeneous, $dm = \rho dxdydz$ (ρ being a constant quantity), and the expression for the moment of inertia becomes

$$\rho\int\int\int(x^2 + y^2)dxdydz = \rho\int\int(x^2 + y^2)(z' - z'')dxdy,$$

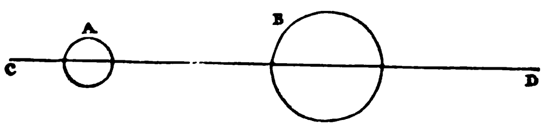
z' and z'' being the two values of z derived from the equation of the surface.

CHAPTER VI.

OF IMPACT.

WHEN one body strikes another, the form of each is altered at the point of impact, by the force with which they press on one another, the change being greater or less according to the constitution of the bodies. A body is said to be perfectly hard if it suffer no change of form whatever; but such perfect hardness does not exist in any known body. It is also found, that most bodies have a tendency to recover their original form as soon as the compressing force ceases to act; this tendency is called elasticity, and a body is said to be perfectly elastic, if in regaining its original form it exerts a force equal to that by which it had been compressed; if the elastic force be less than the compressing force, it is said to be imperfectly elastic: the motions of the bodies after impact depend partly on the forms of the bodies, and partly on their degree of elasticity.

In the following propositions it is intended to consider some of the simplest cases of the motions of non-elastic and perfectly elastic spheres.



Prop. 1. If two non-elastic spheres A and B move so that their centres pass with given uniform velocities along the right line CD, from C towards D, the velocity of A being greater than that of B, until A strikes B, to find their velocities after the impact.

When A strikes B, its anterior surface and the posterior surface of B will be compressed, and the force which they exert to resist this compression will evidently increase the

velocity of B, and diminish that of A, until they are equal, and then since the centres of the spheres cease to approach one another the compression will cease, and therefore since the bodies are supposed to be non-elastic, they will cease to exert any force on one another, therefore they will continue to move uniformly with equal velocities. Let a be the velocity of A, and b the velocity of B, before the impact, and let v be their common velocity after impact, then $a - v$ will be the velocity lost by A, and $v - b$ the velocity gained by B; therefore the forces which produced these changes are to one another as $\text{A}(a - v)$ to $\text{B}(v - b)$; but since these are forces arising from the mutual action of the two bodies, they are equal and opposite (Ax. 5), and therefore $\text{A}a - \text{A}v = \text{B}v - \text{B}b$, whence $v = \dfrac{\text{A}a + \text{B}b}{\text{A} + \text{B}}$.

If one of the bodies B had been at rest $b = 0$, therefore $v = \dfrac{\text{A}a}{\text{A} + \text{B}}$.

If the second body B were moving in the opposite direction to A, then its change of velocity would be $v + b$, whence v would be equal to $\dfrac{\text{A}a - \text{B}b}{\text{A} + \text{B}}$, and in this case if $\text{A}a = \text{B}b$, $v = 0$.

Prop. 2. If two perfectly elastic spheres strike as in the last proposition, to find the velocities after impact.

Let a be the velocity of A, and b the velocity of B before impact, then the compression of the two bodies will proceed as in the last proposition until their velocities become equal, at which time the common velocity will be $v = \dfrac{\text{A}a + \text{B}b}{\text{A} + b}$ (Prop. 1); but in the present case the mutual action will not then cease, but each body will exert the same force in regaining its original form which had been exerted in the compression; therefore the entire change in the velocity of each body is double of the change which takes place during the compression, that is, if U, V be the velocities after impact, $a - \text{U} = 2(a - v)$,

$V - b = 2(v - b)$, therefore $U = 2v - a$, $V = 2v - b$; or substituting for v its value $\frac{Aa + Bb}{A + B}$.

$$U = \frac{Aa + B(2b - a)}{A + B}, \quad V = \frac{A(2a - b) + Bb}{A + B}.$$

If the bodies be equal, $A = B$, therefore $U = b$, $V = a$, that is, the bodies change velocities.

If the body B move in the opposite direction, b is to be taken with the negative sign, as in the first proposition, and

$$U = \frac{Aa - B(2b + a)}{A + B}, \quad V = \frac{A(2a + b) - Bb}{A + B}.$$

If in this case $Aa = Bb$, the equations become $U = -a$, $V = +b$, that is, each body will move with equal velocities after and before impact, but in opposite directions.

If B were at rest, $b = 0$, therefore

$$U = \frac{A - B}{A + B}a, \quad V = \frac{2A}{A + B}a,$$

therefore U would be positive or negative according as A was greater or less than B; in this case if $A = B$; $U = 0$, $V = a$, that is, the whole velocity of the striking sphere is transferred to the other.

If there be a number of perfectly elastic and equal spheres at rest, whose centres are in the same right line, and in contact; and if another equal sphere moving in the line of centres strike the first of these spheres, its velocity will be transferred to this first sphere, and in like manner the first will transfer it to the second, and so on to the last, which will move with a velocity equal to that of the striking sphere, while all the others remain at rest.

N. B.—By equal spheres is meant spheres equal in weight, and not necessarily of equal diameter.

If two spheres, either elastic or non-elastic, strike one another obliquely, the velocity of each may be considered as

compounded of two velocities, one in the direction of the line which joins their centres at the time of impact, the other perpendicular to this line, the latter velocities will be unaltered by the impact (friction being supposed to have no effect), and the velocities in the direction of the line of centres may be calculated as in the above propositions.

APPENDIX.

All machines may be considered as consisting of three parts, namely, the part to which the moving power is applied, that which does the required work, and the apparatus by which these two are connected. For the perfection of the first part it is requisite that the moving power be so applied to it as to do all the work which it is capable of performing; its construction depends on the nature of the moving power: it is usually a piston if the moving power be steam; a wheel of peculiar construction if the moving power be a stream of water; a sail if it be wind, &c. The second part must be adapted to the nature of the work to be performed, its perfection consisting in its not only doing the greatest possible quantity of work, but also doing it well; and for the connecting apparatus it is only requisite that, by a proper combination of the mechanical powers already described, the motion of the former part may communicate the required motion to the latter part, with the least possible loss of power from friction or otherwise, for we shall presently see that it is impossible for this part to increase the quantity of work which the machine can do, although, if ill constructed, it may diminish it.

Now, in order to estimate the relative merits of different kinds of machines, it is necessary to have an accurate measure of the work done by each, and to establish general principles by which to estimate and compare the amounts of work done under different circumstances.

For this purpose it is to be observed that a force can do no work unless it produces motion in the body to which it is applied,

and the most convenient measure of the work done is the product of the force multiplied by the distance through which its point of application moves in the direction of the force; thus a force equal to two pounds weight, moving through one foot, is said to do as much work as a force of one pound weight moving through two feet.

It has been already proved (*Dynamics*, Chap. II., Prop. 8) that if a particle m, originally at rest, be acted on by an uniform force f, and if v be the velocity it acquires in moving through the distance s in the direction of the force, v^2 is proportional to $\frac{fs}{m}$, or, adopting the usual algebraic measures of force and velocity, $v^2 = \frac{2fs}{m}$, and multiplying by m, $mv^2 = 2fs$; the quantity mv^2 is called the *vis viva* of the particle m, and fs is the work done by the force f; therefore the *vis viva* produced by the force is twice the work done by it.

The extension of this principle to more complicated motions leads to the following general propositions of very great practical importance.

Prop. 1. If a single particle of matter, moving along any given line, be acted on by an uniform force in a constant direction, the increase or diminution of its *vis viva* is twice the work done by the force.

First. Let the direction of the motion be the same as the direction of the force, then the increase or diminution of the square of the velocity is equal to the square of the velocity which would have been acquired in falling through the same space from a state of rest (Chap. II., Prop. 11); therefore, multiplying each of these quantities by the quantity of matter in the particle, the increase or diminution of *vis viva* is equal to the *vis viva* which would have been acquired in falling through the same space, which has been already proved equal to twice the work done by the force.

Secondly. Let the particle move in the direction of any other given line, either straight or curved, then, supposing the force which acts upon it to be gravity, the distance moved in the direction of the force will be the vertical height of one extremity of the line above the other, therefore the work done by the force, in moving

along the given line, is the same as the work done in falling down this vertical height; but the velocity acquired in moving along the given line is the same as that acquired in falling down the vertical height (Chap. III., Prop. 6), and therefore the *vis viva* is also the same; but the change of *vis viva*, in falling down the vertical height, has been already proved equal to twice the work done by the force, therefore the change of *vis viva*, in moving along the given line, is so also: and the same is evidently true of any other uniform force as well as of gravity.

Scholium. It is evident that the *vis viva* will increase or diminish according as the angle between the directions of the motion and of the force is acute or obtuse.

Prop. 2. If a single particle be acted on by any force, variable either in quantity or direction, the change of *vis viva* is equal to twice the difference between the work done while the force makes an acute angle with the direction of the motion, and that done while the angle is obtuse.

For the entire time of the motion may be conceived to be divided into intervals so short that the force is constant during each interval; then the increase or decrease of *vis viva* in each interval is twice the work done by the force during that interval (Prop. 1), and the total change of *vis viva* is the difference between the sum of all the increments and the sum of all the decrements, therefore it is equal to twice the difference between the work done by the force while it makes an acute angle with the direction of the motion, and that done while it makes an obtuse angle (Sch. Prop. 1).

Scholium. Since the distance moved in the direction of the force is equal to the space actually passed through multiplied by the cosine of the angle contained by the directions of the motion and of the force, and since the cosine of an acute angle is positive, and of an obtuse angle negative, it follows that the work done is positive or negative, according as this angle is acute or obtuse; and the proposition may be more concisely expressed, thus: "The increase of *vis viva* of a single particle, acted on by any force, is equal to twice the work done by that force;" observing that, in estimating the entire work, the negative portion of it is to be subtracted from the positive.

Prop. 3. Forces perpendicular to the direction of a particle's motion make no change in its *vis viva.*

For such forces make no change in its velocity (Chap. III., Prop. 3).

Prop. 4. If several forces act on a particle in motion, the increase of its *vis viva* is twice the sum of the works done by all the forces (the work being considered negative when the force makes an obtuse angle with the direction of the motion).

For the increase of *vis viva* is twice the work done by the resultant of all the forces (Sch. Prop. 2); but each of the forces may be resolved into two, one in the direction of the motion, the other perpendicular to it; and the resultant may also be resolved into two, in the same directions: then, since the components perpendicular to the direction of the motion have no effect on the *vis viva,* they may be neglected; and since the component of the resultant in the direction of the motion is equal to the sum of the components of the separate forces in the same direction (*Statics*, Chap. I., Prop. 8), the work done by this component is the sum of the works done by these components of the other forces, and the increase of *vis viva,* being double of this, is twice the sum of the works done by the given forces.

Prop. 5. If two particles exert any force on one another, and move in any directions whatever, the sum of the works done by this force on the two particles is equal to the product of the force multiplied by the change of distance between the particles.

For since action and reaction are opposite, the mutual action of the particles must be in the direction of the line drawn from one to the other, therefore the work done by this force on each particle is the product of the force multiplied by the distance which this particle moves to or from the other, according as the force is attractive or repulsive; and the sum of the works done on the two particles is the product of the force multiplied by the sum of the distances by which they approach to or recede from one another, that is, multiplied by the change of distance between them.

Cor. 1. It is evident that the work done is positive or negative, according as the change of distance is in the direction of the mutual action, or in the opposite direction.

Cor. 2. If the distance between the particles be invariable (as in the case of a rigid body), the sum of the works done by their mutual actions is always cypher, whatever their motions may be.

Prop. 6. If the particles of a system in motion be connected together in such a manner, that those which exert any force on one another may preserve an invariable distance during the motion, and if this system be acted on by any external forces, the total change of *vis viva* is twice the sum of the works done by these forces, and is independent of the mutual actions of the particles.

For each particle of the system moves evidently in the same manner as if they were unconnected and acted on by forces equal to those which result from their mutual connexion; therefore the increase of *vis viva* of each particle is twice the sum of the works done on it by the external forces and by the action of the remaining particles (Prop. 4); and the total increase of *vis viva* of all the particles is twice the sum of the works done by all the external forces, and by all the mutual actions of the particles on one another: but the sum of the works done by these latter forces is cypher, since they preserve invariable distances from one another (Prop. 5, Cor. 2), therefore the increase of *vis viva* is twice the sum of the works done by the external forces.

Cor. 1. If the external forces make equilibrium they can make no change in the *vis viva* of any of the particles, therefore the sum of the works done by such forces, in any motion of which the system is capable, must be cypher.

Cor. 2. If the sum of the works done in the first instant of every motion of which the system is capable be always cypher, the forces make equilibrium, for if not, they would produce some motion, and therefore some *vis viva*, in the particles, if they were previously at rest; but the *vis viva* of each particle being necessarily positive, their sum cannot be cypher, and the sum of the works done in the first instant of this motion would be equal to one-half of this sum, instead of being cypher as was supposed.

These two corollaries are evidently equivalent to the principle of virtual velocities (*Statics*, Chap. IV.)

Prop. 7. If the relative motion of two particles in a system be destroyed by their mutual action, the *vis viva* of the system will be

diminished by the square of this relative velocity, multiplied by the product of the two particles, and divided by their sum.

First. Let the particles not be acted on by any other forces, and let them move in opposite directions with velocities inversely as their masses, then, since the velocities destroyed by their mutual action are in the same ratio, the relative velocity will not be destroyed till both particles come to a state of rest, therefore the diminution of *vis viva* in this case is the entire *vis viva* of the two particles; but if v be the original relative velocity, and m, m', the masses of the particles, their velocities must have been

$$\frac{m'}{m+m'}\,\mathrm{v} \text{ and } \frac{m}{m+m'}\,\mathrm{v},$$

and their *vis viva*

$$m\frac{m'^2}{(m+m')^2}\,\mathrm{v}^2+m'\frac{m^2}{(m+m')^2}\,\mathrm{v}^2=\frac{mm'}{m+m'}\mathrm{v}^2.$$

Secondly. Let the particles move in any directions whatever, and let their relative velocity be the same as before, the change of *vis viva* produced by their mutual action will be twice the work done by this force, that is, twice the product of the force multiplied by the change of distance between the particles (Prop. 5); but since both the force and the relative velocity are the same as in the preceding case, the change of distance will be also the same, and therefore the work done, and consequently the diminution of *vis viva* is the same in each case, that is, equal to $\frac{mm'}{m+m'}\,\mathrm{v}^2$.

Thirdly. Let the particles be acted on also by the remainder of the system: then the total change of *vis viva* of the system will be twice the sum of the works done by all the forces; but if the mutual action of the particles m and m' did not exist, the change of *vis viva* would have been twice the sum of the works done by the remaining forces, therefore the difference is twice the work done by this mutual action, that is $\frac{mm'}{m+m'}\,\mathrm{v}^2$, as before.

Cor. 1. It is evident that if relative velocity were produced by the mutual action of particles previously united, there would be an equal increase of *vis viva*; also, if the relative velocity were changed by their mutual action, the change of *vis viva* would be the diffe-

rence of the squares of the relative velocities multiplied by the product of the two particles, and divided by their sum.

Cor. 2. Hence it appears that the impact of non-elastic bodies diminishes their *vis viva* by a quantity proportional to the square of their relative velocity perpendicular to the surface of contact, and that the impact of imperfectly elastic bodies diminishes their *vis viva* by a quantity proportional to the difference of the squares of their relative velocities before and after impact, but that the impact of perfectly elastic bodies makes no change of *vis viva*, since their relative velocity is the same after impact as before.

The principles contained in the foregoing propositions are immediately applicable to the theory of machines in motion. In fact, every machine is constructed for the purpose of doing some required work under the action of a given moving power. Whatever be the nature of the work, it may be ultimately resolved into motion opposed by a given resistance. Now if the work done in any time by the moving power be greater or less than the sum of the works done in the same time by all the resistances, the *vis viva* of the machine will increase or diminish; therefore the machine cannot move with uniform velocity unless the work done in each instant by the moving power be equal to the sum of the works done in the same time by all the resistances, including not only the given resistance, but all obstacles to the absolute or relative motion of the parts of the machine, such as impact, friction, resistance of the atmosphere, &c.; whence it appears that the work usefully done by any machine is less than that done by the moving power, and that the perfection of its construction consists in rendering these two quantities as nearly as possible equal, by diminishing the work done by useless resistances. If the machine be made to do more work than that done by the moving power, the *vis viva* will diminish, until, at length, it becomes cypher, and the machine is then brought to a state of rest; for, since the *vis viva* of a system is the sum of quantities essentially positive, they cannot destroy one another, but each of them separately must be cypher. Also if, on account of insufficient supports or otherwise, motion be communicated to external bodies, the *vis viva* thus uselessly expended will diminish that of the machine, and, in order to preserve its uniform velocity, it will be necessary to

diminish the work done by half the amount of this *vis viva*. In many cases the forces which act on the machine are variable, and the work done by the moving power is alternately greater and less than that done by the resistances. When this is the case, the *vis viva* will increase and diminish alternately, and will always return to its original value, when the total work done by the moving power since the beginning of the motion is equal to that done by the resistances,—the change of *vis viva* being always double of the difference between these quantities. If the machinery be light, this change of *vis viva* may require a very considerable change of velocity; but this inconvenience is avoided by the addition of a heavy fly-wheel.

The principal resistances to the motion of machines are forces opposed to the relative motion of their parts, and the loss of *vis viva* arising from such forces is proportional to the diminution of the square of the relative velocity (Prop. 7). Hence impact always lessens the power of a machine by diminishing its *vis viva*, and it should be avoided in all cases where it is not essential to the nature of the work to be done. Also, when two parts of a machine rub against one another, another retarding force is introduced, which has not yet been considered.

The forces which retard the relative motion of two bodies that rub against one another are adhesion and friction. The force of adhesion is opposed to the relative motion of the bodies in any direction whatever: it is proportional to the extent of the surfaces in contact, but it depends also on other circumstances; its laws are not accurately known, and its amount is inconsiderable except in the case of glutinous bodies.

Friction is a force opposed to the relative motion of two bodies, which are held in contact by a compressing force. It depends on the nature of the bodies, and also, in the case of solid bodies, on the compressing force. The friction of fluids is governed by totally different laws.

Friction of solid bodies is of two kinds or degrees, namely, friction of rest and friction of motion.

Friction of rest is that force which prevents the relative motion of bodies which are in contact and at rest, and which are supposed to have no adhesion. It is opposed to every force which tends to

produce relative motion in the direction of the surface of contact, and is always equal to the force to which it is opposed, provided that it do not exceed a certain limit, which, in given bodies, is proportional to the pressure, but is greater in some bodies than in others, and is found also to depend, in some degree, on the time during which they have been in contact.

Friction of motion is that force which retards the relative motion of two bodies which rub against one another: it is always less than the limiting value of the friction of rest, but agrees with it in being nearly proportional to the pressure, and is always directly opposed to the relative motion of each point of contact of the two bodies; it is nearly independent of the velocity, and is much greater in some bodies than in others.

There is another force analogous to friction, and sometimes called by that name, which retards the motion of rolling bodies; it is much less than the friction of sliding bodies, but appears to be governed by nearly the same laws.

The coefficient of friction is that number which expresses the ratio of that force to the pressure. Thus, if f be the coefficient of the friction of rest for any given bodies, and P the pressure at any point where they touch one another, then the force which prevents relative motion at this point cannot exceed fP; and if f' be the coefficient of the friction of motion, the force which would retard their relative motion at the same point is f'P. In some kinds of wood $f = \frac{1}{2}$ and $f' = \frac{1}{3}$. In polished metals the two coefficients are nearly equal, each being about $\frac{1}{7}$.

The effect of the first kind of friction, in modifying the conditions of equilibrium, will appear from the following Propositions:

Prop. 1. If two plane surfaces rub against one another, without any rotatory motion, their friction is proportional to the total pressure, and does not depend at all on the extent of the surfaces.

For the friction at each point is proportional to the pressure at that point, therefore the sum of all the frictions is proportional to the sum of all the pressures; but since the surfaces are plane, the pressures are a system of parallel forces, and since there is no rotatory motion, all points in the surface of contact must move in parallel directions, therefore their friction is also a system of parallel

forces, and the resultant of each of these systems is equal to their sum; but these sums have been proved proportional, therefore the resultant of all the forces of friction is proportional to the resultant of all the pressures.

Cor. It is evident that the same is true if one only of the surfaces is plane.

Prop. 2. To find what force will cause a body to slide on a given plane without rotatory motion.

Let F be the force and θ the angle which its direction makes with a perpendicular to the plane, and f the coefficient of friction; then $F \cos \theta$ is the pressure on the plane, and (since there is no rotatory motion) $fF \cos \theta$ is the limiting value of friction (Prop. 1); but $F \sin \theta$ is the component of the given force which tends to produce motion, therefore motion will be produced if $F \sin \theta$ exceed $fF \cos \theta$, or (dividing by $F \cos \theta$) if $\tan \theta$ be greater than f.

Cor. 1. Hence it appears that the condition that motion should be produced does not at all depend on the magnitude of the force, but only on its direction: the angle θ, whose tangent is equal to f, is called the "limiting angle of friction."

Cor. 2. If the force acted in such a direction as to produce rotatory motion about an axis perpendicular to the plane, then friction would be no longer a system of parallel forces, and its resultant could not be ascertained without knowing the proportions in which the pressure is distributed over the different points of the surface. If there was rotation about an axis not perpendicular to the plane, the motion would be rolling and not sliding, and would depend on the form of the body.

Prop. 3. To find the least force which will move a heavy body up an inclined plane without rotation.

Let δ be the angle which the direction of the given force makes with the inclined plane, and let ι be the inclination of the plane to the horizon, and θ the limiting angle of friction, then if W be the weight of the body, and P the required force, the pressure on the plane will be $W \cos \iota - P \sin \delta$, and (since $\tan \theta$ is the coefficient of friction) the limiting value of friction is $\tan \theta (W \cos \iota - P \sin \delta)$; but the force which tends to move the body up the plane is $P \cos \delta - W \sin \iota$, therefore motion will commence when this force is greater than the

former, or (multiplying each by $\cos\theta$) when $P\cos\delta\cos\theta - W\sin\iota\cos\theta$ is greater than $W\sin\theta\cos\iota - P\sin\theta\sin\delta$, therefore $P(\cos\delta\cos\theta + \sin\delta\sin\theta)$ is greater than $W(\sin\theta\cos\iota + \sin\iota\cos\theta)$, and P is greater than $W\dfrac{\sin(\iota+\theta)}{\cos(\delta-\theta)}$; but since ι and θ are given, this quantity will be least when $\cos(\delta-\theta)$ is greatest, that is, when $\delta=\theta$, in which case P should exceed $W\sin(\iota+\theta)$.

Cor. Hence it appears that the best direction of draught is at an angle with the plane equal to the limiting angle of friction. It is obvious that at the commencement of the motion the tangent of this angle should be the coefficient of the friction of rest; but that afterwards the angle should be diminished until its tangent is the coefficient of the friction of motion.

Prop. 4. To find the least force capable of supporting a given weight on a given inclined plane.

Adopting the same notation as in the preceding Proposition, the limiting value of friction is $\tan\theta(W\cos\iota - P\sin\delta)$, and the force down the plane is $W\sin\iota - P\cos\delta$, therefore the condition of equilibrium is that the latter force should not exceed the former; and, multiplying each by $\cos\theta$, $W\sin\iota\cos\theta - P\cos\delta\cos\theta$ is not greater than $W\sin\theta\cos\iota - P\sin\delta\sin\theta$, therefore P is not less than $W\dfrac{\sin(\iota-\theta)}{\cos(\delta+\theta)}$, and since ι and θ are given, P is least when $\delta=-\theta$, in which case it is equal to $W\sin(\iota-\theta)$.

Cor. If the force P be horizontal, $\delta=-\iota$, therefore $\cos(\delta+\theta) = \cos(\iota-\theta)$, and the least force in this direction which will support the weight is $W\tan(\iota-\theta)$.

Prop. 5. To find the pressure of a bank of loose earth or sand against a vertical wall.

To simplify this problem we may suppose the upper surface of the earth or sand to be horizontal, and of indefinite extent. It is easy to apply the same principle to any other form of the surface. It is necessary for equilibrium that the pressure of the wall against the bank (which is equal and opposite to the required pressure) be sufficient to keep the bank from falling; therefore, in order to ascertain what this force is, it is necessary, first, to consider how it could fall if the wall did not support it. It is evident that this could only

be by a portion of the earth being separated from the rest by an oblique section of the bank, and sliding down the inclined plane made by this section. Now this section may be in any direction whatever, and the force requisite to sustain the sliding mass will depend on the direction of the section; it is therefore necessary to find which of these will require the greatest force to support it; and if the pressure of the wall be equal to this force, it will, *a fortiori*, prevent the bank from yielding in any other direction.

Now let θ be the limiting angle of friction of the earth or sand, and let W be the weight of the sliding mass, and ι the inclination of the section on which it is supposed to slide, then the horizontal pressure of the wall requisite to support it is $W\tan(\iota-\theta)$ (Cor. Prop. 4); but W, the weight of the triangular wedge of earth, is proportional to its bulk, that is, proportional to the tangent of the angle between the section and the wall, and this angle is the complement of ι, therefore the horizontal force is proportional to $\tan(\iota-\theta)\cot\iota$, which is a *maximum* when the complement of ι is equal to $(\iota-\theta)$, and therefore equal to half the complement of θ; the weight of the wedge is proportional to the tangent of this angle, and the horizontal force requisite to sustain it is equal to the weight multiplied by the tangent of the same angle; the earth is, however, partially supported by the friction of the surface of the wall, therefore the pressure of the bank against the wall is less than this.

N. B. It is easy to find the value of θ by observing the steepest inclination at which a bank of the earth or sand will rest without support. It is evident from Prop. 2 that this inclination is equal to θ; also, it appears from the above value of ι that the plane down which the earth has the greatest tendency to slide bisects the angle between the wall and the steepest natural slope of the earth.

Prop. 6. To find what weight acting at the circumference of a wheel will cause it to turn on a cylindrical axis in a circular hole.

Let θ be the limiting angle of friction between the axis and the hole; then it follows from Prop. 2 that the axis will not begin to slide until it has rolled up the circumference of the hole to a point where the inclination of the radius to the vertical is equal to θ; let r be the radius of the axis, and R the radius of the wheel, then the distance of this point from the vertical through the centre of the

wheel is $r \sin\theta$, and its distance from the direction of the weight at the circumference is $R-r\sin\theta$, but the wheel will not begin to turn on this point of support until the moment of the weight at the circumference exceeds the moment of the weight of the wheel itself, that is, until the weight at the circumference exceeds the weight of the wheel multiplied by $\frac{r\sin\theta}{R-r\sin\theta}$.

Prop. 7. To find the accelerating force on a circular cylinder rolling down a plane by its own weight.

The forces which act on this cylinder are, its weight acting at its centre of gravity, and friction acting at its circumference. The former force may be resolved into two; one perpendicular, the other parallel, to the plane: the accelerating force is evidently the difference between the latter component and friction, divided by the mass of the cylinder. Let F be the friction, m the mass of the cylinder, g the accelerating force of gravity, and ι the inclination of the plane; then this value of the accelerating force becomes $g\sin\iota-\frac{F}{m}$. But since the cylinder rolls down the plane, there is no sliding motion at the point of contact, therefore F is not the friction of motion but of rest, which, as has been already stated, is equal to the force, whatever it be, which tends to produce sliding motion. Now, in order to ascertain this force, it is to be observed that the rotatory motion of the cylinder is produced altogether by this friction, and that it is the same as it would be if the centre of gravity of the cylinder were fixed (*Dynamics*, Chap. I., Prop. 4, Cor. 4); therefore if I be the moment of inertia of the cylinder, and r its radius, the acceleration of the angular velocity is $\frac{Fr}{I}$ (*Dynamics*, Chap. V. Prop. 1): but it is evident that the angular velocity multiplied by r is equal to the linear velocity of the centre of gravity of the cylinder, therefore also the acceleration of the angular velocity multiplied by r is equal to the acceleration of the linear motion, that is to say, $\frac{Fr^2}{I}=g\sin\iota-\frac{F}{M}$, whence $F=\frac{gmI\sin\iota}{I+mr^2}$, and substituting this value in the expression for the accelerating force, it becomes

$$\frac{gmr^2\sin\iota}{I+mr^2}.$$

Cor. 1. It is assumed in this Proposition that friction is capable of preventing the cylinder from sliding, or, in other words, that it is equal to $\frac{gm\text{I}\sin\iota}{\text{I}+mr^2}$; but the pressure on the plane is $gm\cos\iota$, therefore, if f be the coefficient of friction, it cannot exceed the limit $fgm\cos\iota$, therefore the condition that the cylinder shall roll and not slide is that $\frac{gm\text{I}\sin\iota}{\text{I}+mr^2}$ be not greater than $fgm\cos\iota$, or that $\tan\iota$ be not greater than $\frac{\text{I}+mr^2}{\text{I}}f$.

Cor. 2. If the cylinder be homogeneous, $\text{I}=\frac{1}{2}mr^2$, therefore the accelerating force is $\frac{2}{3}g\sin\iota$, and the condition that the cylinder shall roll and not slide is that $\tan\iota$ be not greater than $3f$.

PROBLEMS FOR EXERCISE.

1. Two unequal weights, P and Q, are connected by a cord which passes over a fixed pulley, find the accelerating force on each, the inertia of the pulley being neglected.

Ans. $\frac{\text{P}-\text{Q}}{\text{P}+\text{Q}}g$.

2. In the same case find the tension of the cord.

Ans. $\frac{2\text{PQ}}{\text{P}+\text{Q}}g$.

3. If the moment of inertia of the pulley be $\text{M}k^2$ and its radius r, find the accelerating force.

Ans. $\frac{(\text{P}-\text{Q})r^2}{(\text{P}+\text{Q})r^2+\text{M}k^2}g$.

4. In this case find the tension of each part of the cord.

Ans. $\frac{(\text{M}k^2+2\text{Q}r^2)\text{P}}{\text{M}k^2+(\text{P}+\text{Q})r^2}g$, $\frac{(\text{M}k^2+2\text{P}r^2)\text{Q}}{\text{M}k^2+(\text{P}+\text{Q})r^2}g$.

5. A body unrolls itself from a string attached to a fixed point and rolled round the body in a vertical circle whose centre is the centre of gravity of the body; find the accelerating force on the centre of gravity.

Ans. Let r be the radius of the circle, and k^2 the moment of inertia divided by the mass, then the accelerating force is $\frac{r^2}{r^2+k^2}g$.

6. Two weights, P and W, suspended on a wheel and axle, do not make equilibrium, find the accelerating force on each, neglecting the inertia of the wheel and axle?

Ans. Let R be the radius of the wheel and r that of the axle, then the accelerating force on P is $\frac{PR - Wr}{PR^2 + Wr^2} gR$, and on W $\frac{Wr - PR}{PR^2 + Wr^2} gr$.

7. Find the accelerating force if the moment of inertia of the wheel and axle be given?

Ans. Add the moment of inertia to the denominator of each of these fractions.

8. If the wheel and axle revolve on a cylindrical pivot whose radius is ρ, in a circular hole, and if θ be the limiting angle of friction, find the ratio of the weights in the state bordering on motion?

Ans. $P : W :: r \pm \rho \sin\theta : R \pm \rho \sin\theta$.

9. Find the accelerating force in this case, allowing for friction?

Ans. Substitute in the answers to Problems 6 and 7, for R and r, $R \pm \rho \sin\theta$ and $r \pm \rho \sin\theta$, adopting the upper sign if P preponderates, and the lower sign if W preponderates.

THE END.